Dynamics and Control
Preliminary Examination Topics

Specific topics are described in the textbooks indicated below each general topic area.

I. Dynamics

A. Particle and Rigid Body Dynamics (including material covered in AOE 5204)

Meirovitch, Leonard; *Methods of Analytical Dynamics*, McGraw-Hill, Inc., New York, NY, 1970; Ch. 1 - 5

B. Atmospheric Flight Mechanics (including material covered in AOE 3104 & 3134)

<u>Performance</u>: Anderson, John; *Introduction to Flight, 4th ed*, McGraw-Hill, Inc. New York, NY, 1999; Chapter 6
<u>Stability & Control</u>: Etkin, Bernard, and Loyd Duff Reid; *Dynamics of Flight, Stability and Control, Third Edition*, John Wiley & Sons, Inc., New York, NY, 1996; Ch. 4 - 7

C. Space Flight Mechanics (including material covered in AOE 5204)

<u>Astrodynamics</u>: Bate, Roger, Donald Mueller, and Jerry White; *Fundamentals of Astrodynamics* Dover Publications, New York, NY, 1970; Ch. 1, 2 (Secs. 1-9), 3, 4 (Except Secs. 3-5), 7 (Sec. 4)
<u>Spacecraft Dynamics & Control</u>: Hughes, Peter, *Spacecraft Attitude Dynamics*, John Wiley & Sons, Inc., New York, NY, 1986.

II. Control

A. Linear System Theory (including material covered in AOE 5224)

Rugh, Wilson, *Linear System Theory*, *Second Edition*, Prentice Hall, Upper Saddle River, NJ, 1996; Ch. 1 – 15.

B. Linear Optimal Control (including material covered in AOE 5224)

Kwakernaak, Huibert and Sivan, Raphael; *Linear Optimal Control Systems*, Wiley Interscience, (John Wiley & Sons), New York, NY. 1972, Ch. 3-5

C. Nonlinear System Theory (including material covered in AOE 5344)

Khalil, Hassan, *Nonlinear Systems*, *Third Edition*, Prentice Hall, Upper Saddle River, NJ, 2002; Ch. 1-6, 13-14.

AOE PhD Preliminary Written Exam
Dynamics & Control
Fall 2009

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Please answer four (and ***only*** four) of eight questions, as follows:

- Select and solve **two (2) of the first four (4)** problems, which focus on **dynamics**.
- Select and solve **two (2) of the last four (4)** problems, which focus on **control**.

Adhere to the following guidelines in preparing your solutions:

- Start each question on a ***new sheet*** of paper.
- Write only on the ***front*** of each page.
- ***Write your name*** at the top of each page.

Finally, complete and sign the honor code pledge below and submit this completed cover page with your solutions.

I pledge that this assignment has been completed in compliance with the Graduate Honor Code and that I have neither given nor received any unauthorized aid on this assignment

Signature ____

Printed Name ____

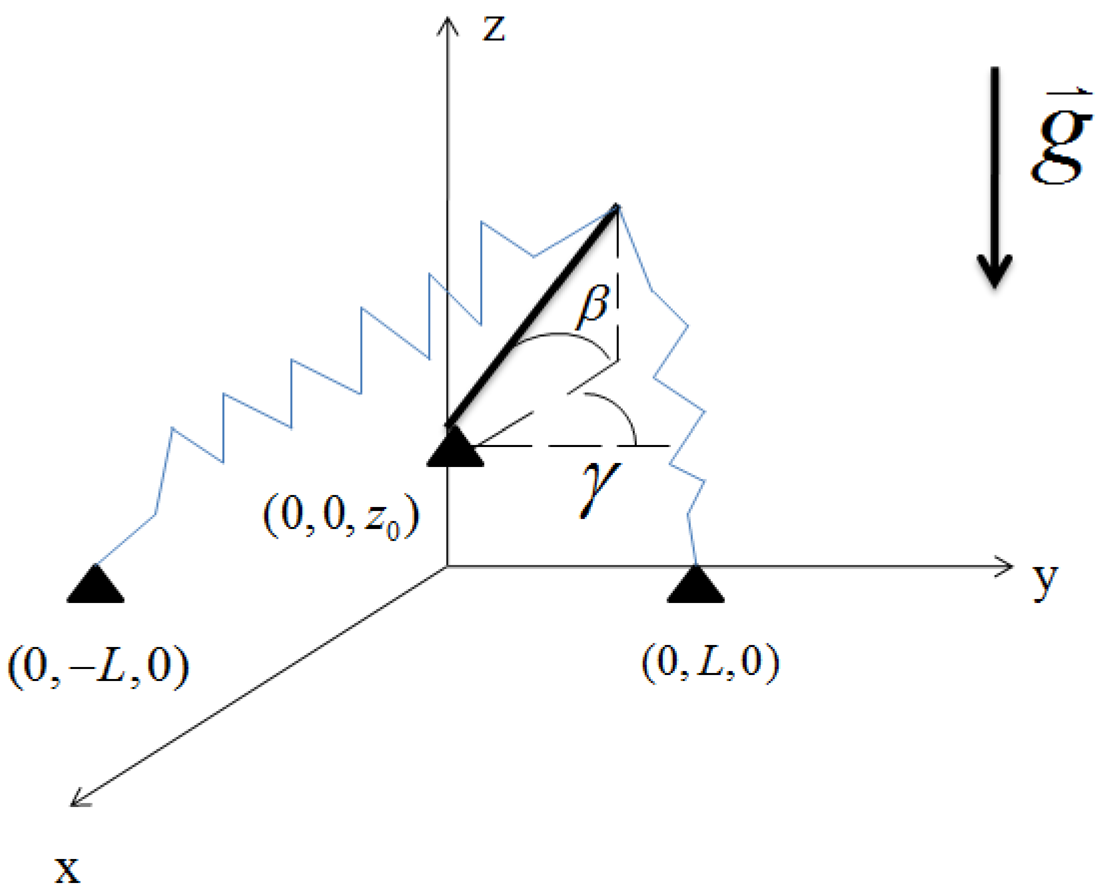
Dynamics Problem #1

1. Dynamics Problem

Consider the system shown in the Figure in which a bar is attached via two identical springs to the ground. The bar can only rotate around its connection point to the ground (i.e. the joint allows rotation by angles β and γ but no translation). Derive the equations of motion for this system using the following assumptions:

- The bar is rigid, of negligible thickness, and it has mass *m*, length *l*, and transversal moment of inertia around the center of mass, *I*.
- The springs are identical, have zero rest-length and spring constant *k*.
- The system is subject to a constant gravitational filed *g* as indicated in the Figure.
- The position of the three joints attached to the fixed ground is indicated in the Figure (i.e. they are all situated in the yz plane and have the indicated coordinates in the Cartesian reference frame fixed to the ground, xyz).
- There is no friction in the system (due to the springs, joints or air) and no other external forces/torques act on the system.

You may use the angles indicated in the Figure (even though that is not a requirement).



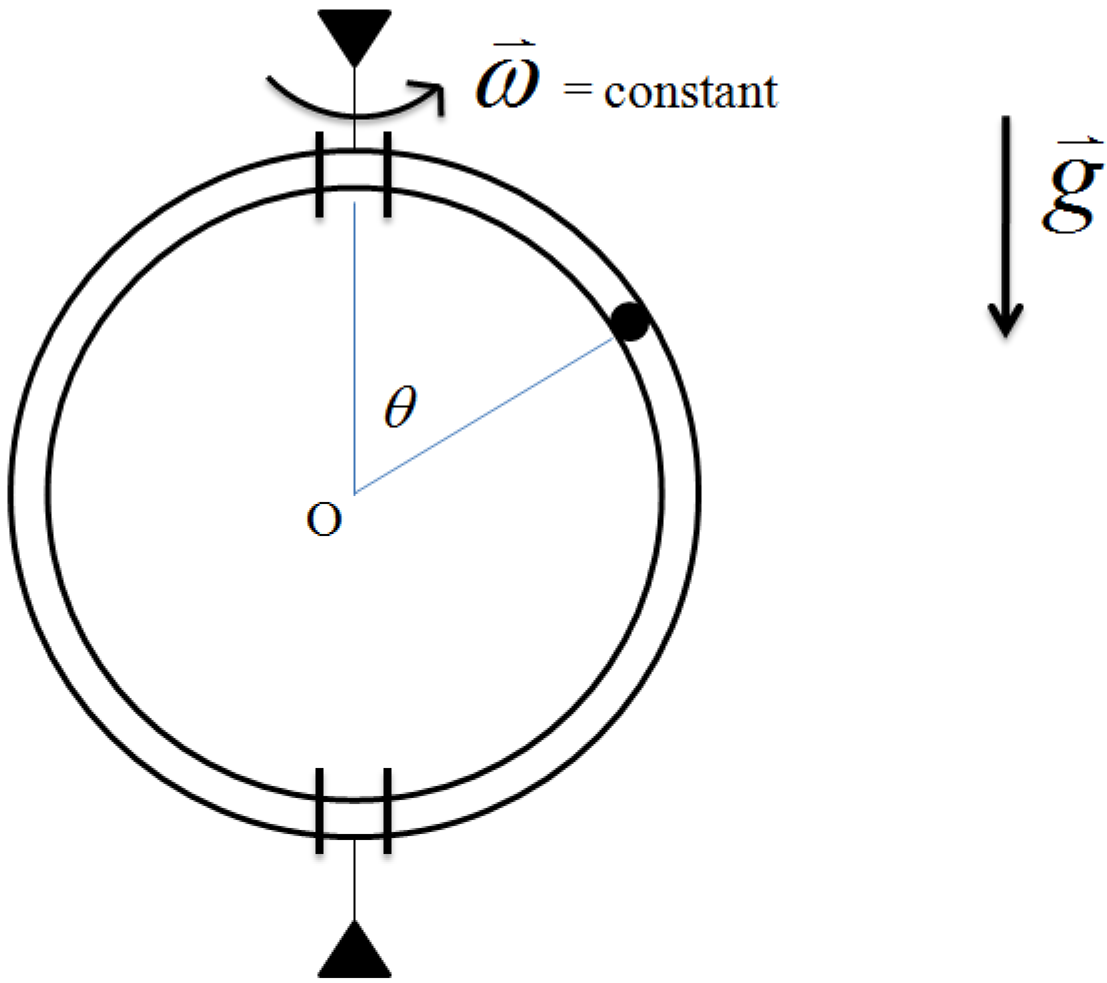
Dynamics Problem #2

2. Dynamics Problem

A point mass of mass m is placed inside a circular tube of radius r. The tube rotates with constant angular velocity ω around the vertical axis. The system is placed in constant gravitational field g, as indicated in the Figure.

a) Derive the equation of motion using the angle θ and assuming that there is no friction in the system (e.g. between the point mass and tube) and the tube is rigid.

b) Integrate the equation of motion to obtain a relation of the form $f(\theta, \dot{\theta}, \theta_0, \dot{\theta}_0) = 0$ where $\theta_0, \dot{\theta}_0$ are initial conditions.



Dynamics Problem #3

AOE Ph.D. Preliminary Examination
Dynamics and Control

Attitude control thrusters are rated according to their "minimum impulse bit," p, in N·s, which is the minimum amount of linear momentum that the thruster can provide in one pulse.

Four thrusters are mounted on a satellite with inertia I, a distance R from the center of mass, as shown in the figure. The thrusters fire whenever the satellite strays from an angular window (the "deadband") of width θ, a technique referred to as "bang-bang" control. More specifically, the thruster logic is defined so that the thruster provides a single impulsive thrust whenever the angular error is outside the deadband and the angular rate is effecting an increase in the angular error. The direction of the thruster firing is selected to decrease the angular error.

Analyze the planar motion of the spacecraft, making reasonable assumptions about initial values of θ and $\dot{\theta}$ as well as disturbance torques. Phase plane (θ, $\dot{\theta}$) sketches for various cases should be included in your analysis.

Dynamics Problem #4

Ph.D. Dynamics and Control Examination
Fall 2009

Figure 1 below shows the empennage (tail section) of an air-to-air missile. The flap at the outer trailing edge of each fin includes a pinned wheel with scoop-shaped teeth that protrude from the outer edge of the flap. The flow of air over these teeth causes each wheel to spin at a high rate. The flap is hinged in the spanwise direction so that a flap deflection results in a roll moment about the longitudinal axis of the missile. The flap is not actuated, however -- this is a passive roll rate stabilization device. Provide a mathematically convincing analysis that verifies the stabilizing effect of this mechanical feedback device.

Figure 1. A photograph of the empennage of a high-speed missile (left) and a simplified illustration (right). Note the mechanical roll stabilization device embedded at the outer, trailing edge of each tail fin.

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Control Problem #1

Control Problems for Ph.D. Preliminary Examination

Control Problem #1. Consider an LTI system subject to a stationary, zero-mean, Gaussian disturbance process whose net effect is as follows:

$$\dot{x} = Ax + Bu + w.$$

The power spectral density of $w(t)$ is W. Suppose also that the following noisy measurements are available:

$$y = Cx + v,$$

where $v(t)$ is a stationary, zero-mean, Gaussian white noise process with power spectral density V. Assume that A is skew-symmetric, ***i.e.***, $A^T = -A$ and that (A, B) is controllable and (A, C) is observable. Then, given $W = C^T C$ and $V = I$ (identity matrix), find an output feedback LQG controller for this system which renders the closed-loop system stable and minimizes

$$\lim_{t \to \infty} E\left\{x^T B B^T x + u^T u\right\}.$$

Note that you need to verify first that the problem is well-posed in the sense that a stabilizing LQG solution exists.

Control Problem #2

Control Problem #2. Consider the linear time-varying system

$$\dot{x}(t) = \underbrace{\begin{bmatrix} -1 & 0 \\ e^{3t} & -2 \end{bmatrix}}_{A(t)} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t).$$

(1) Find the state transition matrix $\Phi(t, t_0)$ corresponding to $A(t)$.

(2) Find the eigenvalues of matrix $A(t)$ for all $t \geq 0$. Based on this information *only*, can you deduce whether the system is stable or not?

(3) Suppose that the initial state $x(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ and the control input $u(t) = 0$ for all $t \geq 0$. Do the system states remain bounded as $t \to \infty$?

(4) Suppose the control input is applied in discrete-time with a sampling period T, specifically u will be constant over each time interval $[kT, (k+1)T)$ for all non-negative integers k, i.e.

$$u(t) = u_k \qquad \text{for all} \quad kT \leq t < (k+1)T.$$

Then, using zero-order hold sampling, we obtain the following discrete-time state-space difference equation:

$$x_{k+1} = \bar{A}_k \, x_k + \bar{B}_k \, u_k,$$

where $x_k = x(kT)$ and $u_k = u(kT)$ for all integers $k \geq 0$. Compute $\bar{A}_k$ and $\bar{B}_k$.

Control Problem #3

Ph.D. Dynamics and Control Examination
Fall 2009

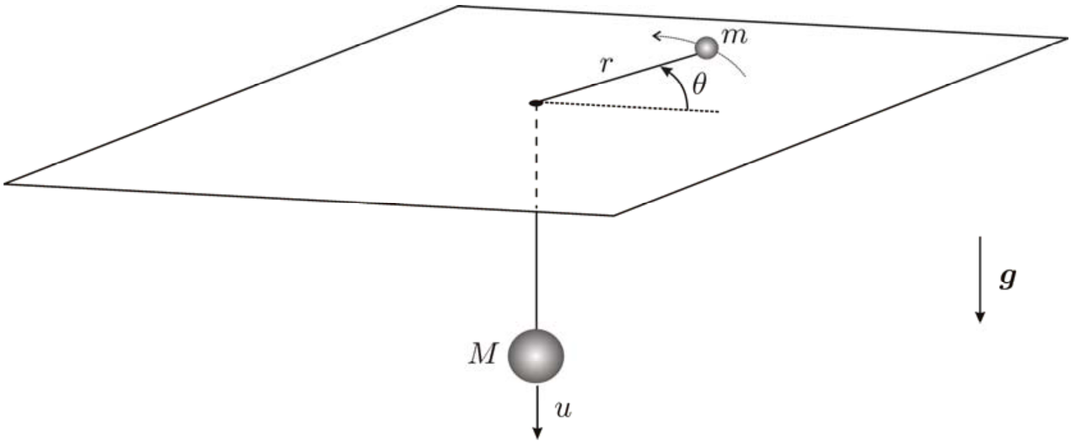

Figure 1. A mechanical control system.

Consider two point masses m and M which are connected by an inextensible cord that passes through a small hole in a horizontal plate. The smaller mass m slides along the plate while the larger mass M hangs vertically. A control force u acts on M in the vertical direction as shown. Assume there is no friction in the system.

1. Derive the equations of motion using your favorite technique.
2. Compute the rate of change of energy and angular momentum.
3. Assume a constant (nonzero) value for the angular momentum and find conditions for a circular orbit with $u = 0$. Linearize the radial dynamics about the circular orbit that you found and compute the eigenvalues of the linear state matrix. Comment on the stability of this "relative equilibrium."
4. Assume that r and $\dot{r}$ are available for feedback and compute the control law u which minimizes

$$J = \frac{1}{2}\int_0^\infty Ru(\tau)^2 d\tau$$

where R is a positive, scalar constant. Comment on stability of the desired circular orbit under this choice of feedback.

Control Problem #4

Ph.D. Dynamics and Control Examination
Fall 2009

For the following two planar dynamical systems, determine through appropriate analysis whether the equilibrium at the origin is

- Uniformly asymptotically stable
- Globally uniformly asymptotically stable
- Exponentially stable
- Globally exponentially stable

System 1.

$$\begin{aligned}\dot{x}_1 &= (\sin\omega t)\, x_2 - \left(1+\cos^2\omega t\right) x_1^3 \\ \dot{x}_2 &= -(\sin\omega t)\, x_1 - \left(1+\cos^2\omega t\right) x_2^3\end{aligned}$$

where ω is a positive, constant parameter.

System 2.

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \|\boldsymbol{x}\|^2 \begin{pmatrix} \cos\omega t \\ \sin\omega t \end{pmatrix}$$

where $\boldsymbol{A}$ is a constant, Hurwitz matrix and ω is a positive, constant parameter.

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AOE Ph.D. Preliminary Examination
Dynamics and Control

Aerobraking is a passive orbit control concept useful for placing a spacecraft into a circular orbit about a planet with a sensible atmosphere (such as Venus, Earth, and Mars). The desired circular orbit radius will typically be larger than the radius of the planet plus the sensible atmosphere. The idea is to use active control to place the spacecraft into a highly elliptical orbit with periapsis inside the sensible atmosphere and apoapsis well outside the atmosphere. During the periapsis passage for each orbit, the spacecraft will lose orbital energy due to drag, thereby decreasing the energy (and therefore the semimajor axis) of the orbit. Since the spacecraft moves fastes near periapsis, a useful first approximation assumes that the periapsis altitude does not change and that the spacecraft experiences an impulsive Δv at periapsis, thereby reducing the velocity at periapsis, which reduces the orbital energy. Once the apoapsis radius has been lowered to the radius of the desired circular orbit, a short-duration rocket motor is fired to circularize the orbit in a nearly impulsive maneuver.

The objective of this problem is for you carefully to develop an algorithm for determining how many orbits it will take to complete the aerobraking portion of of this maneuver. You must explicitly state all assumptions, define all variables, and explain how the algorithm would be implemented in a standard programming language. You might use a pseudo-code to accompany this explanation.

Finally, explain how the size of the circularizing rocket motor would be determined and how the point of firing the circularizing rocket would be chosen.

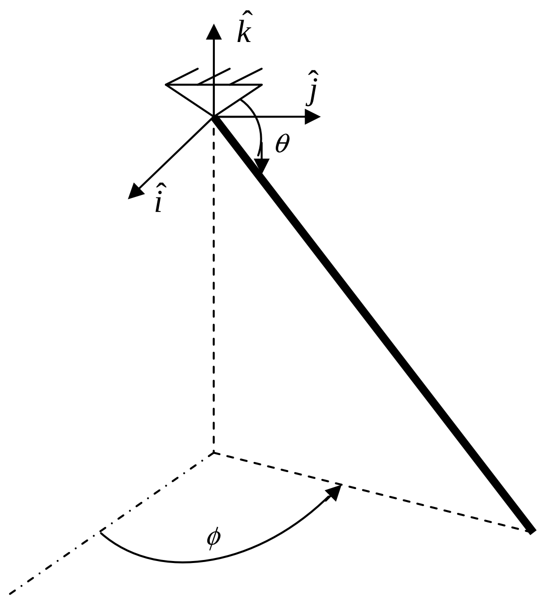
Dynamics Problem #2

AOE Ph.D. Preliminary Examination
Dynamics and Control

A rigid rod of zero thickness, constant, uniformly distributed mass m and length l is connected by a frictionless spherical joint to a fixed point O. The rod is subject to constant gravitational acceleration g in the vertical (pointing downward) direction and to a velocity dependent potential which is expressed, using the angles indicated in the Figure, as $V = -\frac{1}{2}C(l\sin\theta)^2\dot{\phi}$ where C is constant.

A. Find the equations of motion.

B. Given initial conditions $\theta_0,\ \phi_0,\ \dot{\phi}_0$ find an expression for $\dot{\phi}$. (*Hint*: Use an "integral of the motion" provided by the Lagrangian.)

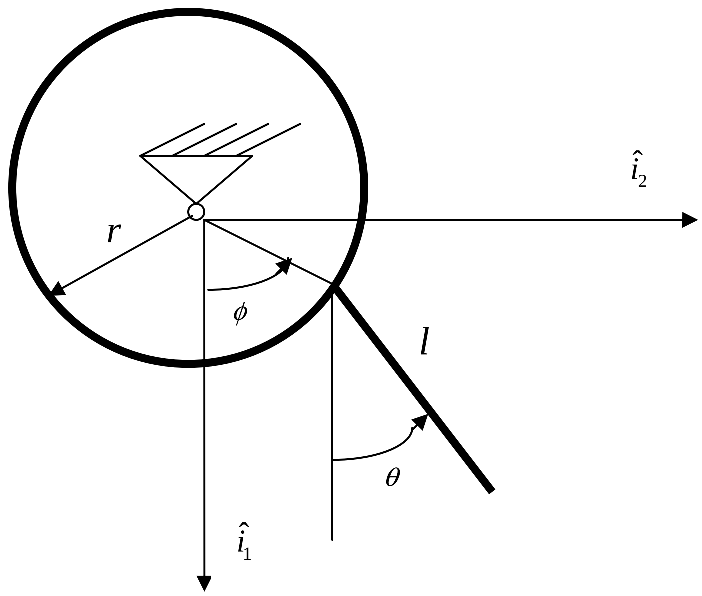


Dynamics Problem #3

AOE Ph.D. Preliminary Examination
Dynamics and Control

A. Derive the equations of motion for a system composed of a disk and a rod showed in the Figure. Use the following assumptions:
  - the disk is rigid, has radius r, uniformly distributed mass M, and moment of inertia I around the axis passing through its center of mass and perpendicular to the disk;
  - the disk rotates freely in the vertical plane around its center of mass, which is fixed as indicated in the Figure.
  - the rigid rod has length l, zero thickness, uniformly distributed mass m
  - the rod rotates freely in the vertical plane around the point of attachment to the disk.
  - there is no friction in the system and the gravitational field is constant, pointing downward.

B. Linearize around the equilibrium position $\phi = 0,\ \ \theta = 0$ (*Hint*: Use the second order form of the equations of motion.)

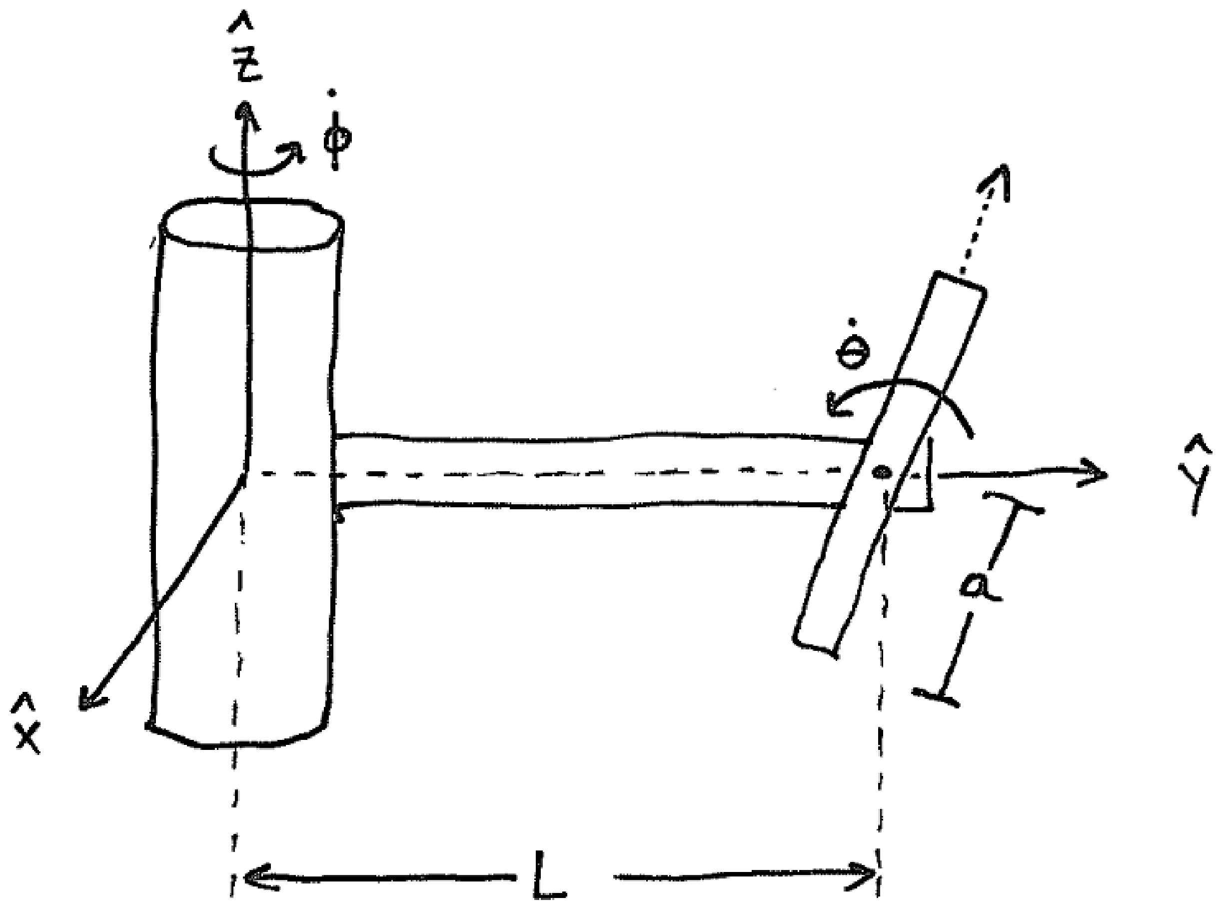


Dynamics Problem #4

AOE Ph.D. Preliminary Examination
Dynamics and Control

A) Calculate the kinetic energy and the angular momentum about the origin of the uniform thin bar of mass per unit length, ρ. The bar is of length 2a and is symmetric about the attachment point. Consider the mass of the horizontal bar to be negligible and let the connection be frictionless. There are no external forces on the system.

B) Now, let $\dot{\theta}$ be caused by an external torque τ about the pivot point of the thin bar. Derive the equations of motion using Lagrange's method.



Control Problems for Ph.D. Preliminary Examination

Control Problem #1. Consider the system equation

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t) + k(t), \qquad x(t_0) = x_0,$$

where $k(\cdot)$ is a *known* n-dimensional vector function defined on the finite interval $[t_0, t_f]$, with continuous entries. Suppose that the performance index is

$$V = \int_{t_0}^{t_f} \left(x^T Q(t) x + 2u^T N(t) x + u^T R(t) u\right) dt + x^T(t_f)\, M\, x(t_f),$$

where all the matrices have continuous entries, and

$$R(t) \succ 0, \qquad Q(t) - N^T(t) R^{-1}(t) N(t) \succeq 0, \qquad M \succeq 0.$$

(i) Using the *Hamilton-Jacobi-Bellman equation*, show that the optimal control for this problem can be written as

$$u^*(t) = -R^{-1}(t) \left(\left(B^T(t) P(t) + N(t)\right) x + B^T(t) h(t) \right),$$

where $P(\cdot)$ is a matrix function ($\succeq 0$) and $h(\cdot)$ is a vector function. Obtain the expressions (differential equations) satisfied by P and h.

(ii) Obtain an expression for the minimum value of V.

Control Problem #2

Control Problem #2.

(a) Consider the linear time-varying system:

$$\dot{x} = A(t)\,x, \qquad y = C(t)\,x,$$

where $A(t)$ and $C(t)$ are continuous functions of t. Prove or disprove the following statement. (If the statement is false, then producing a counterexample will suffice.)

- *In general*, if the pair $(A(t), C(t))$ is observable over some time interval $[t_0, t_f]$, then $(-A(t), C(t))$ is also observable over $[t_0, t_f]$.

(b) Consider the linear time-invariant system: $\dot{x} = Ax$, $y = Cx$, where A and C are constant matrices. Suppose that this system is detectable.

(i) Show that if the output $y(t) = 0$ for all t along some state trajectory $x(t)$, then this trajectory must satisfy: $x(t) \to 0$ as $t \to \infty$.

(ii) Show that there exists a positive definite matrix Q such that the function $V(x) = x^T Q x$ satisfies $\frac{dV}{dt} < y^T y$.
Hint: There exists a matrix E such that $H + EF + F^T E^T$ is negative definite if and only if there exists a real scalar α such that $H - \alpha F^T F$ is negative definite.

Control Problem #3

Ph.D. Dynamics and Control Examination
Fall 2010

Consider the nonlinear system:

$$\dot{\boldsymbol{x}} = \begin{pmatrix} -x_1^3 + x_2 \\ x_1^2 - x_2 \end{pmatrix}$$

A. Find all equilibria.

B. Provide analysis to fully characterize the stability of each equilibrium. That is, determine if each equilibrium is unstable, stable, (globally) asymptotically stable, (globally) exponentially stable. (*Hint*: You may want to consider the flow of the vector field in a region bounded by the two curves $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$.)

Control Problem #4

Ph.D. Dynamics and Control Examination
Fall 2010

Figure 1 below shows a suspended pendulum whose pivot point is subject to a time-varying horizontal acceleration $a(t)$, a continuous, bounded function of time. The pendulum is equipped with a servo-motor that applies a torque T about the pivot. The motor bearing exerts a damping moment that is linear in the pendulum angular rate.

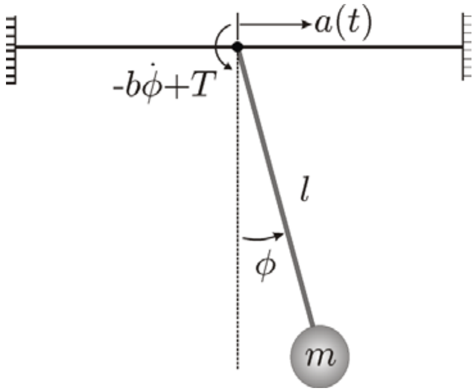


Figure 1. A suspended pendulum with a horizontally perturbed pivot point.

The parameters of the system are uncertain, but they satisfy the following bounds (in SI units):

$$0.5 \le m \le 1.5, \qquad 0.9 \le l \le 1.1, \qquad 0 \le b \le 0.2, \qquad \text{and} \qquad |a(t)| \le 1$$

1. Derive an ordinary differential equation that describes the pendulum's motion. (Don't treat the pivot position as a configuration variable; treat the horizontal acceleration $a(t)$ as a disturbance.)
2. Design a state feedback control law which ensures that the state components satisfy the following ultimate bounds:

$$|\phi(t)| \le 0.01 \text{ rad} \qquad \text{and} \qquad |\dot{\phi}(t)| \le 0.01 \text{ rad/s}$$