Studies of Plasma Instabilities using Unstructured Discontinuous Galerkin Method

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Abstract

The discontinuous Galerkin (DG) method is employed in this work to study plasma instabilities using high-order accuracy. DG method has the advantage of resolving shocks and sharp gradients that occur in neutral fluids and plasmas. An unstructured DG code has been developed in this work to study plasma instabilities in general geometries using the two-fluid plasma model. Unstructured meshes are known to produce small and randomized grid errors compared to traditional structured meshes. Benchmark tests for Euler and MHD system are performed. MHD and preliminary two-fluid plasma model results are provided for simulation of Kelvin-Helmholtz instabilities.

Discontinuous Galerkin Method

DG method combines features of the finite element and finite volume methods and it is only piecewise continuous (Cockburn and Shu [1988]). When using the DG method, we project the solution onto a set of orthogonal bases. The polynomial bases can be either modal bases or nodal bases as shown in equation 1 and 2.

Modal form:

$$u_h^k(x,t) = \sum_{i=0}^{N} \hat{u}_i^k(t)\psi_i(x) \tag{1}$$

Nodal form:

$$u_h^k(x,t) = \sum_{i=1}^{Np} u_i^k(x_i^k,t)\ell_i^k(x) \tag{2}$$

Then we multiply both sides of the equations with the orthogonal test functions and integrate them over one cell. For 2-D hyperbolic equations in balance-law form we have:
Conservation Law:

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} = 0 \tag{3}$$

Here we choose Lagrange polynomial as the test function. Then the problem becomes cell-wise. Excluding the source term on the right hand side, we solve the following equation for each cell, and update the solution using Runge-Kutta methods.

Nodal DG representation:

$$\int\int_{I_k} \left(\frac{\partial \mathbf{q}}{\partial t}\ell_i - \mathbf{F}\frac{\partial \ell_i}{\partial x} - \mathbf{G}\frac{\partial \ell_i}{\partial y}\right)dxdy + \int_{\partial I_k}((\mathbf{F}n_x + \mathbf{G}n_y)^*\ell_i)ds \tag{4}$$

For more details about RKDG methods see Cockburn and Shu [1989] and Hesthaven and Warburton [2007].

Unstructured DG code

An unstructured DG code is developed to perform the simulations for arbitrary geometries. This code uses nodal DG with Lax-Friedrichs numerical flux. High order numerical method often results in numerical oscillations for solutions with sharp gradients. These oscillations can violate positivity of density and pressure for problems with low densities or low pressure. Thus, a slope limiter developed by Moe et al. [2015] is applied to reduce the numerical oscillations on the shock locations and a positivity-preserving limiter from Zhang et al. [2012] is extended to maintain positive density and pressure for the MHD and two-fluid equations.

Benchmark cases for Euler and MHD systems

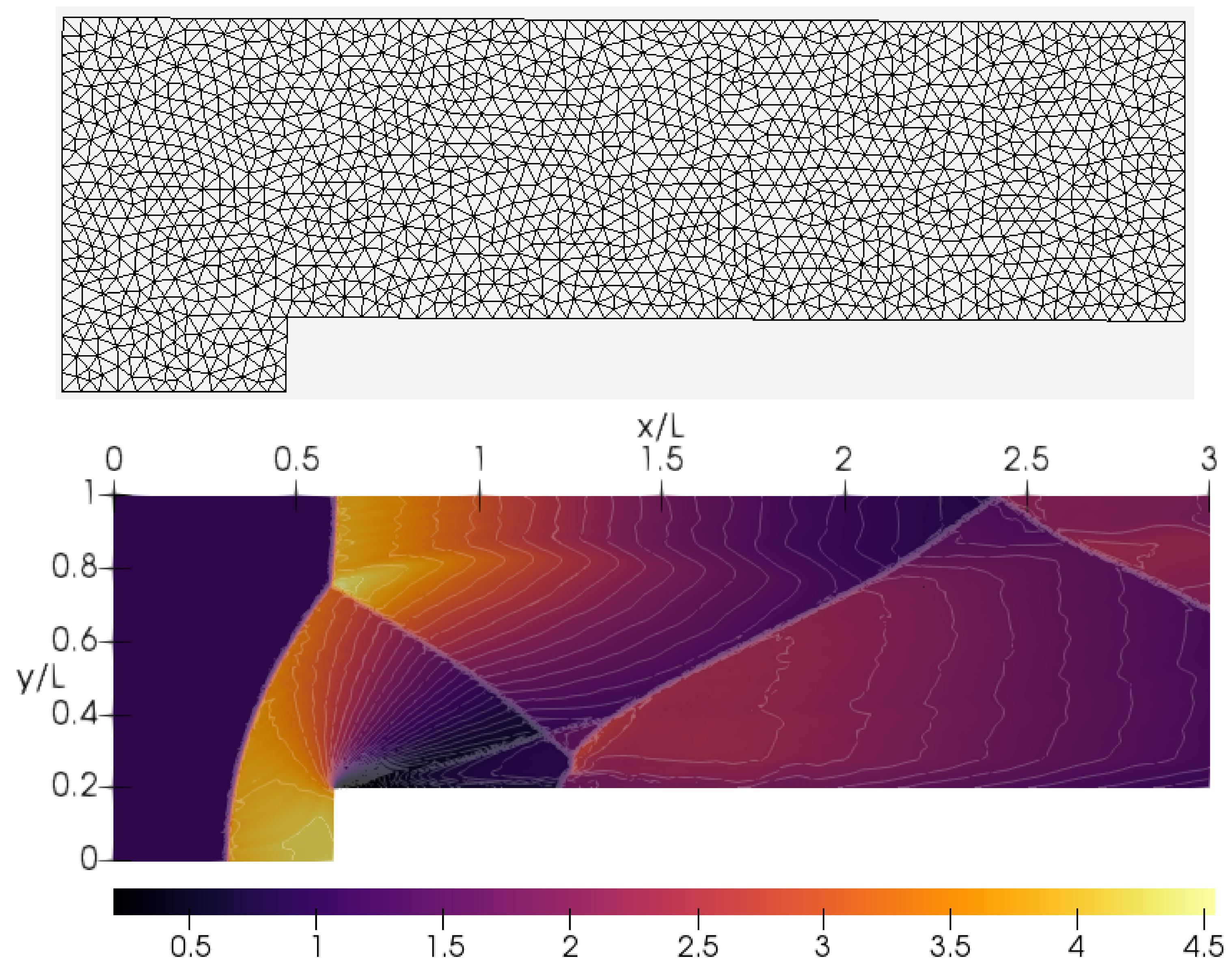

Figure 1: Top: demonstration of the unstructured mesh used. Bottom: density profile of forward facing step case at time $t = 4.0$ using Euler's equation. Mesh resolution: 67,000 trangular elements.

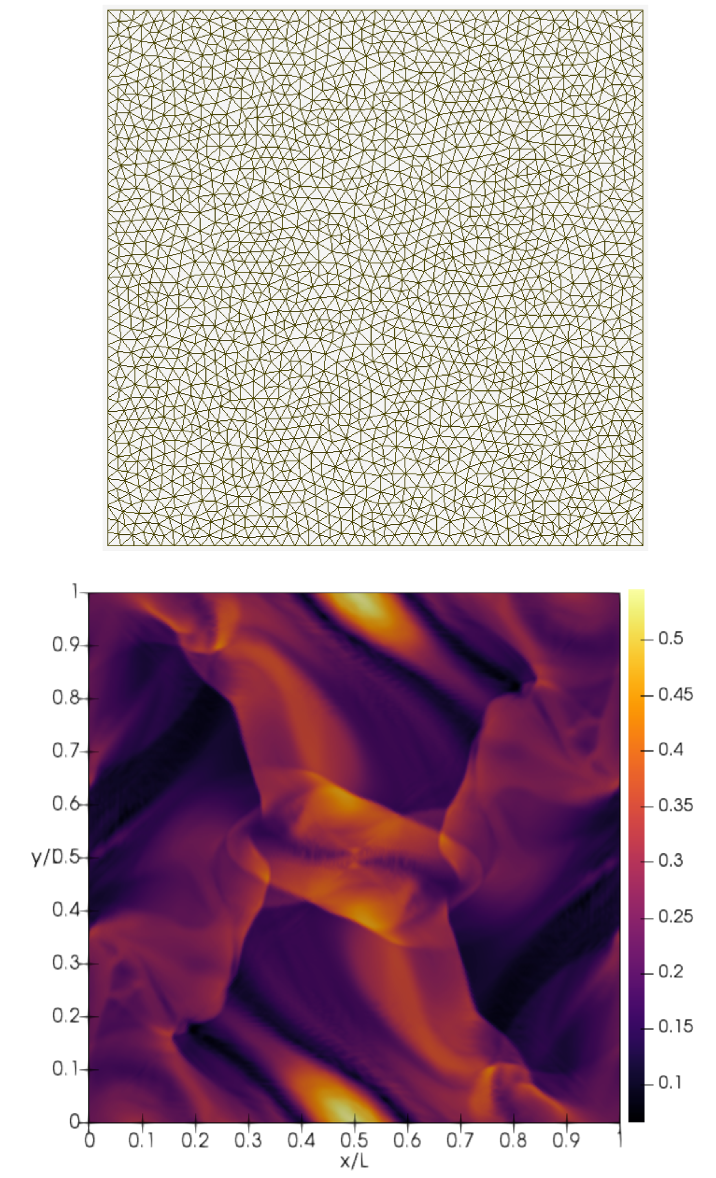

Figure 2: Top: demonstration of the unstructured mesh used. Bottom: density (ion) profile of Orszag-Tang vortex at time $t = 0.5$ using ideal-MHD model. Mesh resolution: 27,000 trangular elements.

Forward facing step initial and inflow boundary condition $\left((\rho, u, v, p) = (1.0, 3.0, 0.0, \gamma^{-1})\right)$ (Mach 3 inflow) is used to benchmark the Euler solver. Figure 1 shows the late-time density profile. For the MHD case, the Orszag-Tang vortex is a well-known, well-benchmarked problem in plasma turbulence. This case tests the robustness of the unstructured DG algorithm for the MHD equations when there are significant interactions between shocks. Figure 2 shows the late-time density evolution of the vortex. For both cases, second-order polynomial Runge-Kutta Discontinuous Galerkin method is used.

Kelvin-Helmholtz instability

The Kelvin-Helmholtz instability (KHI) occurs in a range of plasmas from fusion to space. Benchmark simulations are performed using the unstructured discontinuous Galerkin method applied to the MHD and two-fluid plasma models with a code presently under development. Initial conditions include regions of different velocity, the inner fluid has a density of $\rho = 2$ and the outer fluid has a density of $\rho = 1$. A uniform magnetic field is initialized in the x-direction. A single mode perturbation is applied initially on the velocity in the y direction. Boundary condition is periodic everywhere in the domain. Figure 4 presents late-time density evolution of a single-mode KHI for the MHD and the two-fluid models. Also included is a plot of the out-of-plane B_z magnetic field which is zero for the MHD model but non-zero for the two-fluid model due to currents that develop. The ion-to-electron mass ratio is chosen to be 100 in the two-fluid plasma simulations.

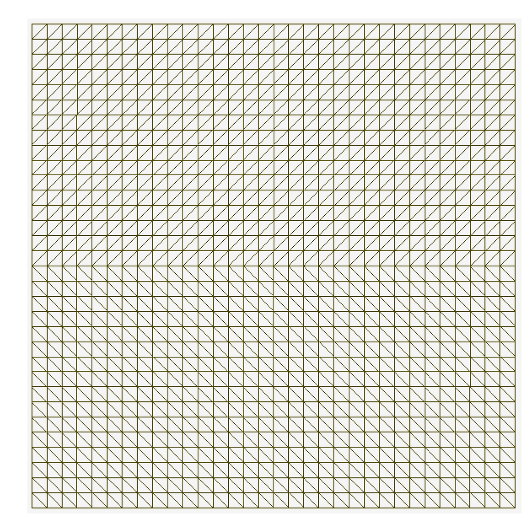

Figure 3: Grid selected for KHI simulations

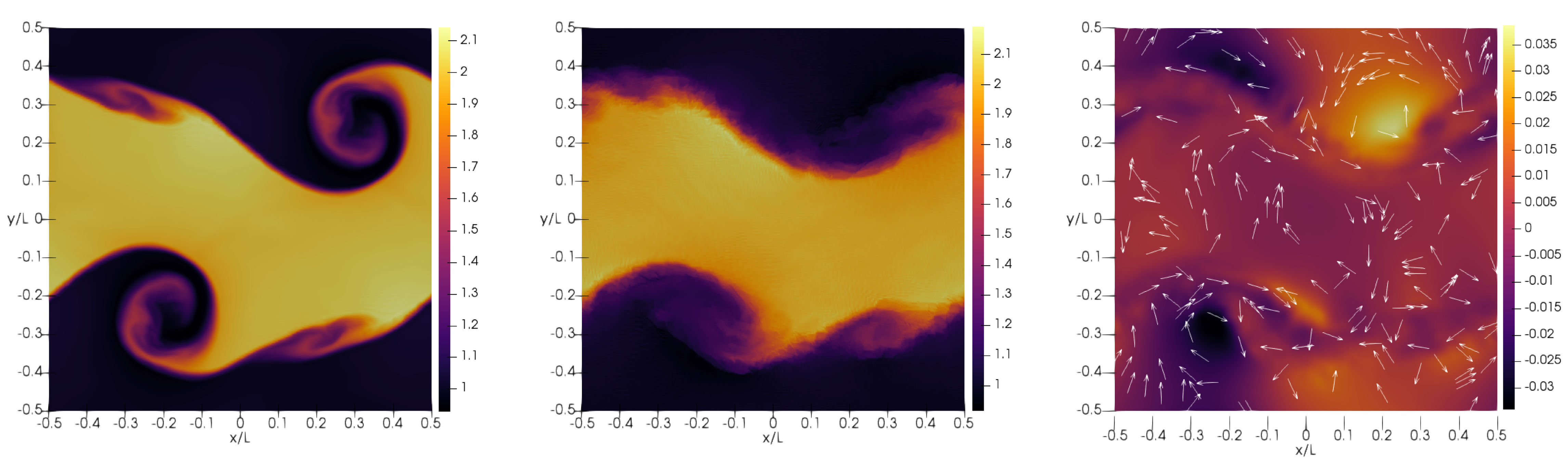

Figure 4: Density (ion) profile of Kelvin-Helmoholtz instability at time $t = 2.5$ using ideal-MHD model (left) and two fluid plasma mode (middle). The out-of-plane magnetic field (B_Z) from the two-fluid model is also shown (right). A mesh with 133,000 triangular elements is used in these simulation with a second-order polynomial basis for the Runge-Kutta Discontinuous Galerkin method. Vectors of the current are superimposed on B_z.

MHD model

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0\\
&\frac{\partial \rho\mathbf{u}}{\partial t} + \nabla\cdot\left(\rho\mathbf{uu} + p\mathbf{I} - \frac{\mathbf{BB}}{\mu_0} + \frac{B^2}{2\mu_0}\mathbf{I}\right) = 0\\
&\frac{\partial \epsilon}{\partial t} + \nabla\cdot\left[\left(\epsilon + p + \frac{B^2}{2\mu_0}\right)\mathbf{u} - \frac{(\mathbf{B}\cdot\mathbf{u})}{\mu_0}\mathbf{B}\right] = 0\\
&\frac{\mathbf{B}}{\partial t} + \nabla\cdot(\mathbf{uB} - \mathbf{Bu}) = 0
\end{aligned} \tag{5}$$

Two-fluid plasma model

$$\begin{aligned}
&\frac{\partial \rho_s}{\partial t} + \nabla\cdot(\rho_s\mathbf{u}_s) = 0\\
&\frac{\partial \rho_s\mathbf{u}_s}{\partial t} + \nabla\cdot(\rho_s\mathbf{u}_s\mathbf{u}_s + p_s\mathbf{I}) = \frac{\rho_s q_s}{m_s}(\mathbf{E} + \mathbf{u}_s\times\mathbf{B})\\
&\frac{\partial \epsilon_s}{\partial t} + \nabla\cdot[(\epsilon_s + p_s)\mathbf{u}_s] = \frac{\rho_s q_s}{m_s}\mathbf{u}_s\cdot\mathbf{E}\\
&\epsilon_s \equiv \frac{p_s}{\gamma - 1} + \frac{1}{2}\rho_s u_s^2
\end{aligned} \tag{6}$$

We solve Euler's equation for each species and then solve Maxwell equations.

Summary

An unstructured high-order discontinuous Galerkin code is developed and benchmarked for cases using Euler and MHD equation systems. Computations with plasma Kelvin-Helmholtz instabilities are presented using MHD and two-fluid plasma model. Further tests and benchmarks for the two-fluid plasma solver are underway.

Acknowledgements

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