

AOE 4144: Applied CFD

A series of 12 lectures by Prof. Raj (course co-instructor)

Reflections on the Effectiveness of Applied Computational Aerodynamics for Aircraft Design

Pradeep Raj, Ph.D.

Collegiate Professor Emeritus

Kevin T. Crofton Department of Aerospace and Ocean Engineering

Virginia Tech, Blacksburg, Virginia, USA

<http://www.aoe.vt.edu/people/emeritus/raj.html>

Program Management Director, Lockheed Martin (Retired)

Deputy Director, Technology Development & Integration

The Skunk Works®, Palmdale, California, USA



LOCKHEED MARTIN



Lecture 2

Topic 2: Genesis of Fluid Dynamics

List of Topics

Preface

1. Introduction

2. Genesis of Fluid Dynamics (*Antiquity to 1750*)

3. Fluid Dynamics as a Mathematical Science (1750–1900)

4. Emergence of Computational Fluid Dynamics (1900–1950)

5. Evolution of Applied Computational Aerodynamics (1950–2000)

5.1 *Infancy through Adolescence (1950–1980)*

Level I: Linear Potential Methods (LPMs)

Level II: Nonlinear Potential Methods (NPMs)

5.2 *Pursuit of Effectiveness (1980–2000)*

Level III: Euler Methods

Level IV: Reynolds-Averaged Navier-Stokes (RANS) Methods

6. ACA Effectiveness: Status and Prospects (2000–20xx)

6.1 *Assessment of Effectiveness (2000–2025)*

6.2 *Prospects for Fully Effective ACA (Beyond 2025)*

7. Closing Remarks

Appendix A. An Approach for ACA Effectiveness Assessment

Why Look Back?

***Study the past, if you would
define the future.***

— Confucius (551 – 479 BC)



***The further backward you look,
the further forward you can see.***

— Churchill (1874 – 1965)

The Old Testament (1200 – 165 BC)

Proverbs 30:18-19

“There be three things which are too wonderful for me, yea four which I know not.”

“The way of an eagle in the air, the way of a serpent upon a rock, the way of a ship in the midst of the sea, and the way of a man with a maid.”

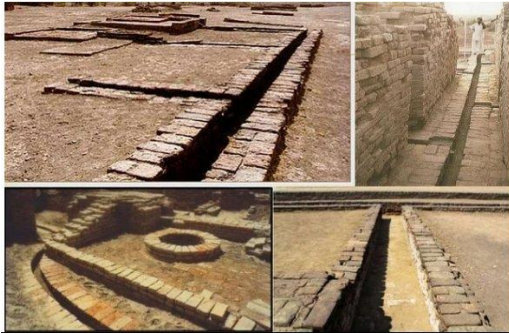


Two of The Three Things Involve Flow of Fluids and They Remain “Too Wonderful” Today!

Early Days of Civilization

Two Crucial Needs

1. **Water Distribution to villages and cities for farming and household use**—canals and conduits were built to transport water



Indus valley water supply
(ca 2350-1900 BC)

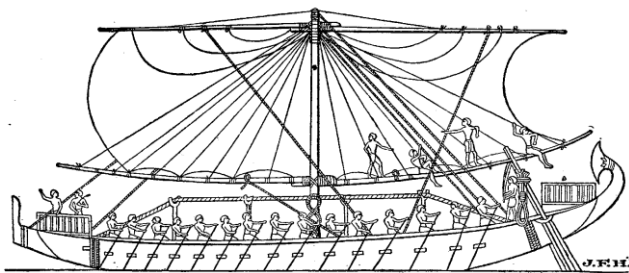


Eupalinos underground aqueduct
(ca 600 BC)



Aqua Anio Vetus Roman
aqueduct (ca 272 BC)

2. **Maritime Transport to supply essential goods**—river boats and seafaring ships powered by sails or manual propulsion were built

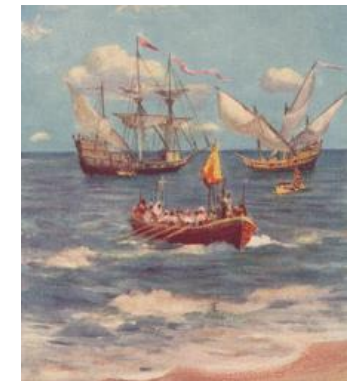


Egyptian ship on the Red Sea, about 1250 B.C. [From Torr's "Ancient Ships."]
Mr. Langton Cole calls attention to the rope truss in this illustration, stiffening the beam of the ship. No other such use of the truss is known until the days of Modern engineering.

Ancient Egyptian ship (ca 1250 BC)



Vikings landing in Britain (ca 449 AD)



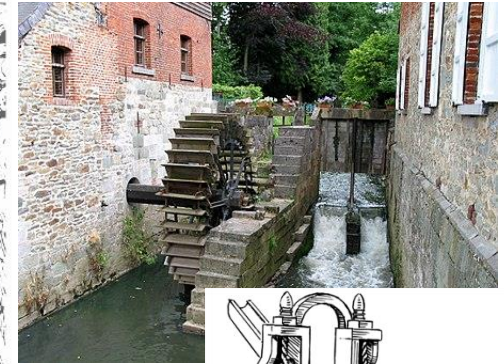
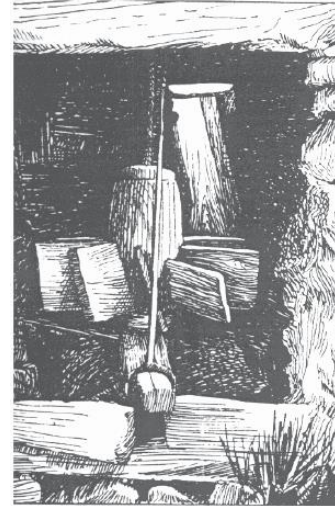
Vasco da Gama at
Calicut, India (ca 1498 AD)

Early Days of Civilization

Two Sets of “Grand Challenge” Problems

1. Problems of Resistance

- Motivating societal needs:
 - navigation (ships)
 - fluid-driven machines (waterwheels and mills)
 - ballistics (projectiles)
- *How does a fluid current affect a body in its path?*



2. Problems of Discharge

- Motivating societal needs:
 - water distribution
 - jet reaction machines
- *How do fluids discharge themselves from reservoirs and through tubes or pipes?*



Impetus Behind the Genesis of “Fluid Dynamics”

Addressing “Grand Challenge” Problems

Two Branches of Investigations Emerged

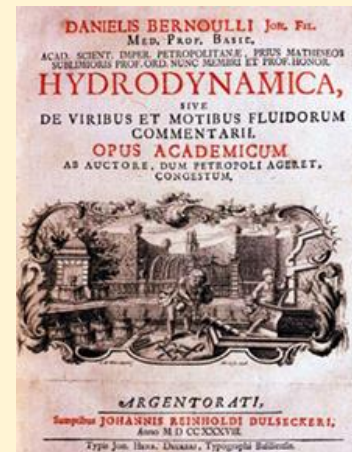
Hydraulics

Artisan Activity based on Empirical Knowledge to devise Practical Solutions to problems of fluids in motion or at rest

Flourished for Countless Millennia

Hydrodynamics

Scientific Activity based on Laws of Nature to develop Fundamental Understanding and Knowledge of fluid flow



Daniel Bernoulli



8 Feb 1700 – 17 Mar 1782

Formally Emerged in 1738

Evolved as the preferred approach to solve fluid flow problems!

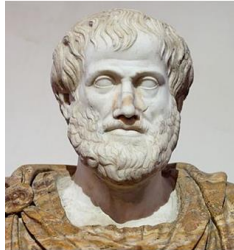
Key Foundational Theories, Principles, and Laws of Fluid (Aero/Hydro) Dynamics

Antiquity to 1750

Concept of Continuum

Medium Theory

Aristotle



384–322 B.C.

Scientific Observation of Flows

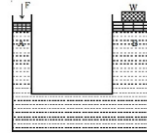
Leonardo da Vinci



15 April 1452 – 2 May 1519



Pascal's Law



Blaise Pascal



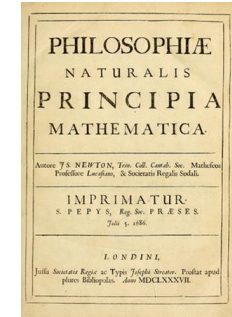
19 Jun 1623 – 19 Aug 1662

Laws of Mechanics of Motion

Isaac Newton



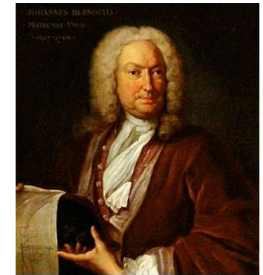
25 Dec 1642 – 20 Mar 1727



Book 2:
The Motion of Bodies
(in Resisting Mediums)

"Internal Pressure" in Moving Fluids

Johann Bernoulli



6 Aug 1667 – 1 Jan 1748

Antiquity

1400

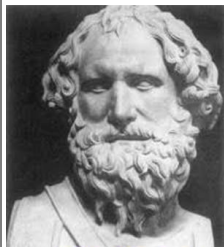
1500

1600

1700

1750

Archimedes



287–212 B.C.

Archimedes' Principle
Hydrostatics

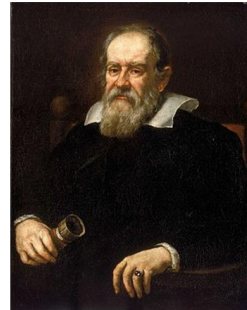
Simon Stevin



1548 – 1620

Stevin's Principle

Galileo Galilei



15 Feb 1564 – 8 Jan 1642

Scientific Method

Evangelista Torricelli



15 Oct 1608 – 25 Oct 1647

Torricelli's Law

$$v \propto \sqrt{h}$$

Christiaan Huygens



14 Apr 1629 – 8 Jul 1695

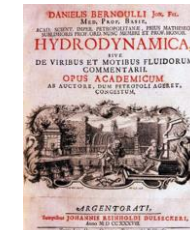
Huygens' Law

$$R \propto V^2$$

Daniel Bernoulli



8 Feb 1700 – 17 Mar 1782



Jean le Rond d'Alembert



16 Nov 1717 – 29 Oct 1783

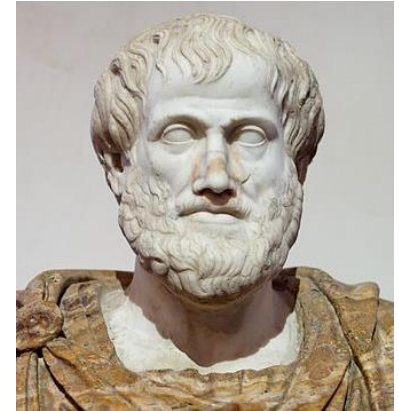
'Theoria Resistancia' d'Alembert's Paradox

In the Beginning... *The Greek Ideas*

- **Four Basic Elements Theory**

- Universe consists of four basic elements: *fire, air, water, earth*
- Protagonists included Pythagoras (~580-500 BCE), Empedocles (490-430 BCE), Plato (427-347 BCE), and Aristotle (384-322 BCE)
 - *Their theories significantly departed from mythology*
- **Aristotle—a pure theorist—probably had the most influence on the growth of scientific knowledge in general, and fluid mechanics in particular**, that lasted nearly 2,000 years

Aristotle



Greek Philosopher
384–322 BCE

- **Nature Abhors Vacuum**

- Space around us must be occupied by one element or another
- **Vacuums—the absence of any and everything—were simply an impossibility.**

- **Concept of Continuum**

“The **continuous may be defined as that which is divisible into parts which are themselves divisible to infinity**, as a body which is divisible in all ways. Magnitude divisible in one direction is a line, in three directions a body. Being divisible in three directions, a body is divisible in all directions. And magnitudes which are divisible in this fashion are continuous.”

- **Theory of Motion**

- In a void, a body at rest will remain at rest, and a body in motion will continue to have the same motion unless some obstacle comes into collision
- Everything that is in motion must be moved by something. **A body in motion is being driven by fluid closing in behind.** [An arrow creates a vacuum in its wake, into which air rushes, pushing it from behind.] *Paradoxically, air also resists motion!*

“The Birth of Hydrostatics”

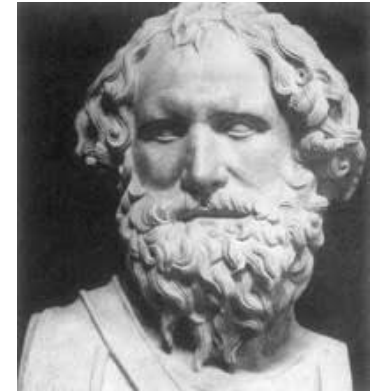
Basic Principles

- **Proposition 3:** Of solids those which, size for size, are of equal weight with a fluid will, if let down into the fluid, be immersed so that they do not project above the surface but do not sink lower.
- **Proposition 4:** A solid lighter than a fluid will, if immersed in it, not be completely submerged, but part of it will project above the surface.
- **Proposition 5:** Any solid lighter than a fluid will, if placed in the fluid, be so far immersed that the weight of the solid will be equal to the weight of the fluid displaced.
- **Proposition 7:** A solid heavier than a fluid will, if placed in it, descend to the bottom of the fluid, and the solid will, when weighed in the fluid, be lighter than its true weight by the weight of the fluid displaced.
- **Postulates:** Fluids cannot have internal empty spaces, i.e., they *must be continuous*. And if fluid parts are continuous and uniformly distributed, then that which is the least compressed is driven along by that which is more compressed. In a fluid “**each part is always pressed by the whole weight of the column perpendicularly above it.**”

Archimedes' Principle (or Law)

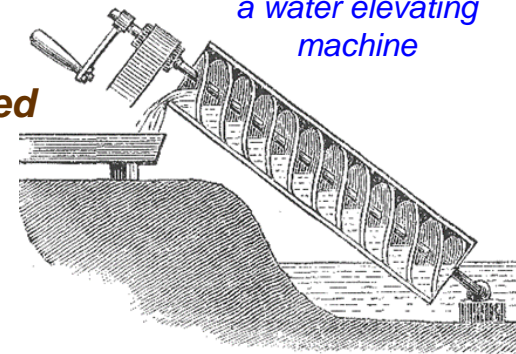
When a solid body is immersed in a fluid, it is pressed vertically upwards by the fluid with a force equal to the weight of the fluid displaced, the force is known as buoyancy.

Archimedes



Greek Mathematician
287–212 BCE

Archimedes' Screw a water elevating machine



[Arguably] No Major Advancements for Next 17 Centuries!

Direct Study of Nature: *The Renaissance* L2 (15th Century)

- da Vinci: The *First Scientific Observer of Flows*



- **Principle of Continuity**

“By so much as you will increase the river in breadth, by so much you will diminish the speed of its course.” (i.e., $\text{area} \times \text{speed} = \text{constant}$)

- **Principle of Relative Motion**

The air's action is the same whether the bird is at rest in a moving airstream—hovering at a cliff edge in a strong breeze—or is moving through still air.

- **Principle of Circulation**

“The helical or rather rotary motion of every liquid is so much the swifter as it is nearer to the center of its revolution...the motion of the [solid] circular wheel is so much the slower as it's nearer the center...[for water] we have the same motion, through speed and length, in each whole revolution of the water, just the same in the circumference of the greatest circle as in the least...”

- **Air Resistance is Directly Proportional to Speed**

Leonardo da Vinci



Italian Artist, Engineer, Scientist
15 Apr 1452 – 2 May 1519

Scientific Method: *The Renaissance* (16th Century)

- **Emergence of Scientific Method**

Galileo adds Experimentation and Quantification to Da Vinci's Observation for studying nature

- **Tenets of Scientific Method**

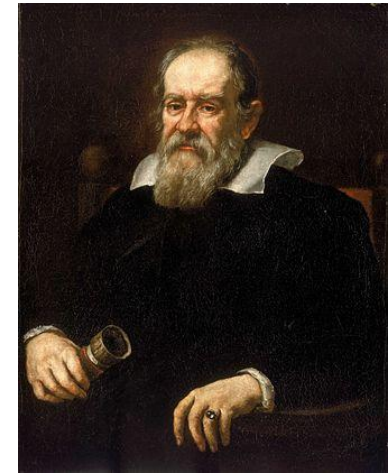
- **OBSERVE:** Observe phenomena
- **HYPOTHESIZE:** Formulate *hypotheses* via induction
- **TEST:** Experimentally test deductions from hypotheses
- **REFINE:** Use findings to *refine* or *eliminate hypotheses*

- **Galilean Principle of Inertia**

"A body in motion would remain in motion unless a force caused it to come to rest."

It contradicted the widely accepted Aristotelian theory of motion

Galileo Galilei



Italian Philosopher, Astronomer and "Geometer" (Mathematician)
15 Feb 1564 – 8 Jan 1642

"Philosophy is written in this grand book, which stands continually open before our eyes (I say the 'Universe'), but cannot be understood without first learning to comprehend the language...it is written in mathematical language, and its characters are triangles, circles and other geometric figures..."

-- Galileo Galilei, The Assayer, Oct. 1623

Advancement of Hydrostatics (16th Century)

Simon Stevin

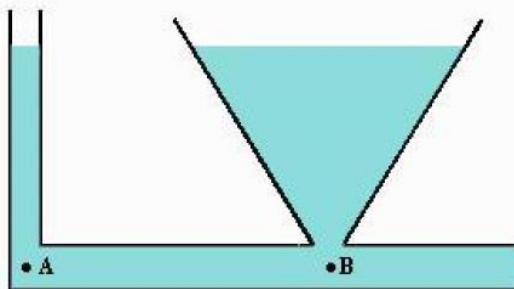


Flemish Mathematician
1548 – 1620



- Genesis of 'Hydrostatic Paradox'**

The **hydrostatic pressure** at the bottom of a container filled with a liquid depends, linearly, only on the height of the liquid column, and not on the particular shape (and thus on the volume) of the container

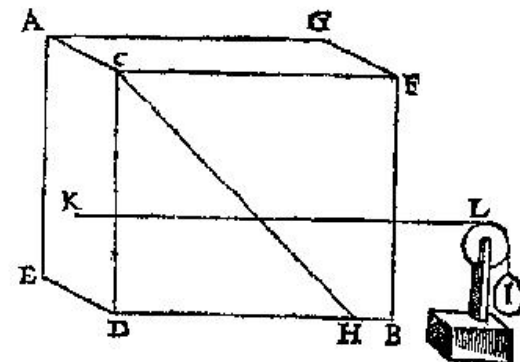


- Principle of Solidification**

In any fluid *at rest*, if any portion be replaced by a rigid solid, the forces exerted by the remainder will not be altered

- Pressure on the Side of a Vessel**

Used limit arguments to prove that water in the rectangular box exerts a force at the center of mass of the vertical wall ACDE equal to that of the weight of the water volume ACHDE



“Any column of water, however small, may be made to support any weight, however large.”

First Notable Contributions Since Archimedes!

Advancement of Fluid Statics (17th Century)

L2

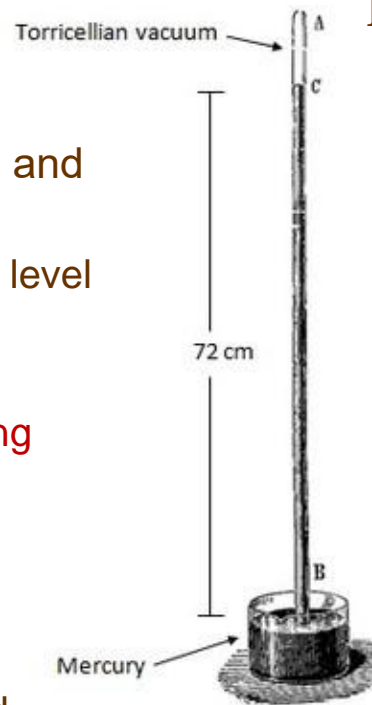
• Barometric Pressure

- **Torricelli (1630)** invents mercury barometer; gives partial explanation of its operation
- **Pascal (1647)** repeats Torricelli's experiment, and further studies atmosphere
 - Variation of atmospheric pressure cause liquid level to change from day to day
 - Atmospheric pressure reduces with altitude
 - *"Nature does not abhor vacuum" -- contradicting prevailing Aristotelian wisdom*
- Pascal proves that pressure at any point in a fluid is the same in all directions

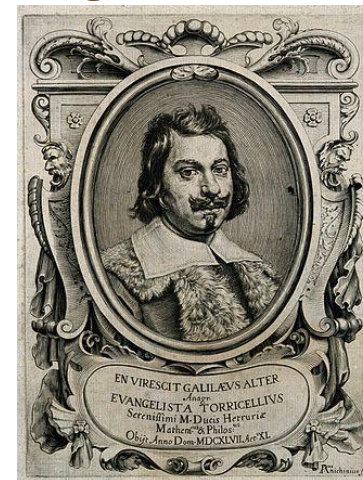
• Pascal's Law (1647-48)

- A change in pressure at any point in an enclosed fluid *at rest* is transmitted undiminished to all points in the fluid
- Resolves Hydrostatic Paradox, and enables development of hydraulic devices

"In order to show that a hypothesis is evident, it does not suffice that all the phenomena follow from it; instead, if it leads to something contrary to a single one of the phenomena, that suffices to establish its falsity."
-- Blaise Pascal



Evangelista Torricelli

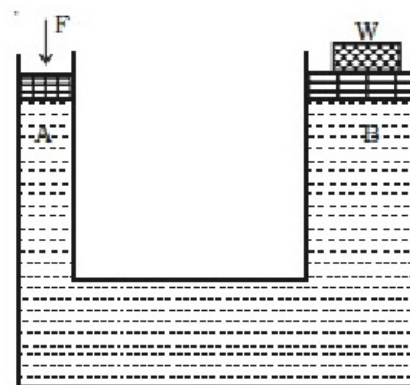


Italian Physicist
15 Oct 1608 – 25 Oct 1647

Blaise Pascal



French Philosopher
19 Jun 1623 – 19 Aug 1662



Study of Discharge Problem (17th Century)

L2

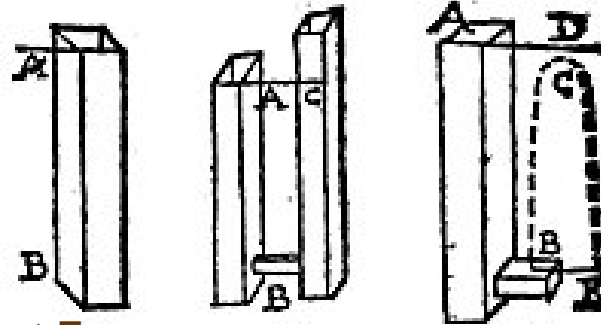
Efflux of Water from Vessels

Torricelli's Law (1644)

Efflux velocity is proportional to the square root of the depth: $v \propto \sqrt{h}$

Water jet from a small hole rises *almost* to the same height as the water level in the tank.

The upwards velocity at B is the same as the downwards velocity at E.



Study based on application of Galilean principle for falling motion of bodies to the efflux of liquids from vessels!

Huygens Experiments (1668)

Confirmed Torricelli's Law!

However, disparate results obtained based on the geometry of the apparatus, such as, form of the vessel, type of spout, relative location of orifice to the surface of the vessel.

Modified Law: $v \propto k\sqrt{h}$

Proportionality constant, k, adjusted to match measurements!

Evangelista Torricelli



Italian Physicist
15 Oct 1608 – 25 Oct 1647

Christiaan Huygens



Dutch Scientist
14 Apr 1629 – 8 Jul 1695

Study of Resistance Problem (17th Century)

Resistance of Fluid on Bodies

Huygens Law (1669)

“Resistance is proportional to the square of the fluid velocity”
(when the velocity doubles, the resistance quadruples)

- Deduced from experiments with projectiles.

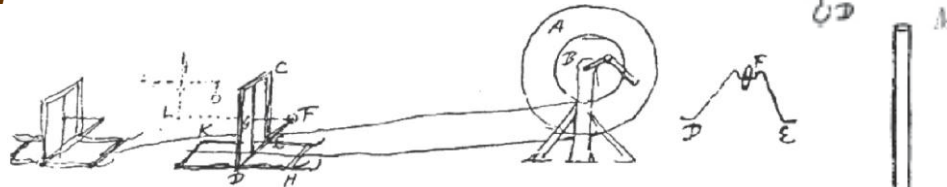
Corrects prevailing thought that resistance is proportional to the fluid velocity (when the velocity doubles, the resistance doubles)

Experimentally measures resistance of

(i) a wooden cube being dragged through a water channel



(ii) fully submerged bodies moving through air



Christiaan Huygens



Dutch Scientist
14 Apr 1629 – 8 Jul 1695

Edme Mariotte

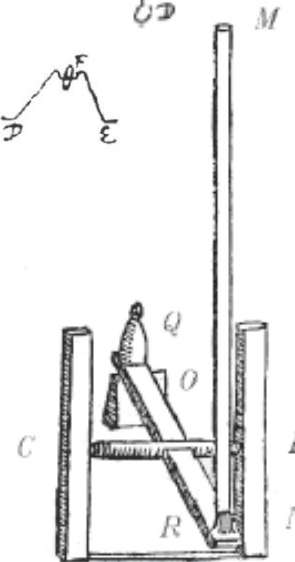


French Physicist
1620 – 12 May 1684

Mariotte's Principle (1673)

Resistance is proportional to the square of the fluid velocity (when the velocity doubles, the resistance quadruples)

- Deduced from experiments with moving fluid impacting on a flat surface.



Insights into the Nature of Fluids (17th Century)

L2

- Boyle's Hypothesis (1661)**

Matter consists of little particles in motion; every phenomenon is the result of *collisions of particles in motion*.

- Two types of Fluids: Liquids (water) & Gases (air)**

Liquids form a free surface not created by their container;
Gases occupy the entire volume of the container.

- Boyle's Law (1662)**

"The product of pressure (P) and volume (V) is a constant for a given mass of confined gas as long as the temperature is constant."

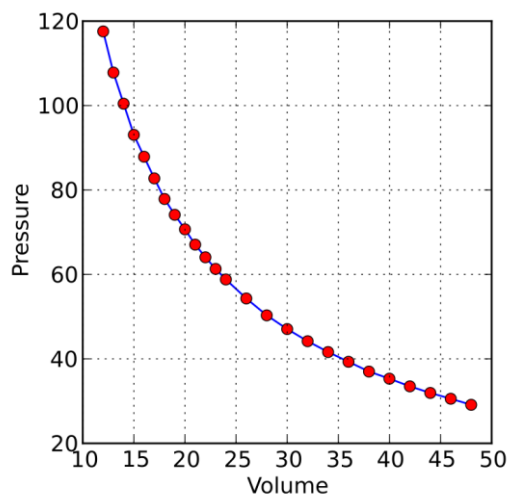
$$P \propto \frac{1}{V}$$

Liquids may be regarded as incompressible.

- Charles' Law (1780—a century later)**

"The volume (V) of a gas increases linearly with the absolute temperature (T) of the gas as long as pressure is constant."

$$V \propto T$$



Robert Boyle



Anglo-Irish Philosopher
25 Jan 1627 – 31 Dec 1691

Jacques Charles



French Physicist
12 Nov 1746 – 7 Apr 1823

Basic Laws of Mechanics of Motion (17th Century)

L2

- **Mathematical Principles of Natural Philosophy (July 5, 1687)**

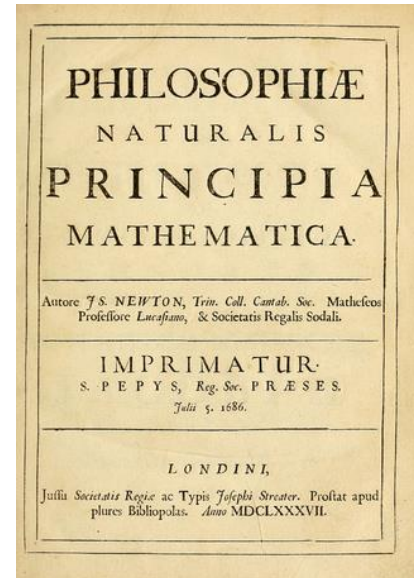
“...the basic problem of [natural] philosophy seems to be to discover the forces of nature from the phenomena of motions and then to demonstrate the other phenomena from these forces; and to this end the general propositions in the first and second Books are directed.”

- **Book I: Of the Motion of Bodies**

- Deals with rigid bodies (point masses)
- ***First complete, rational, theoretical derivation of all motions from a few axioms and laws***

- **Book II: Of the Motion of Bodies (in Resisting Mediums)**

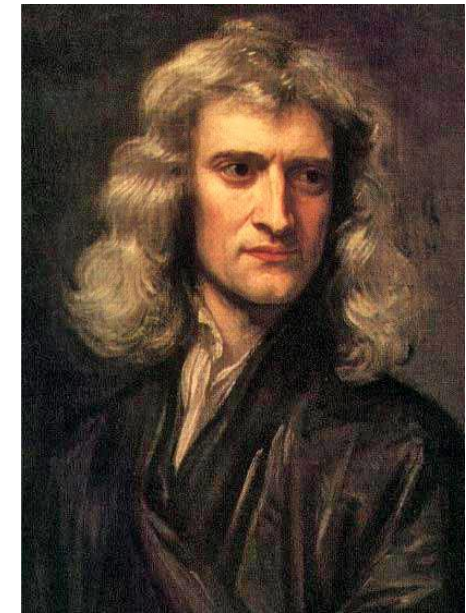
- Deals with two types of fluid models:
 - ‘Rare Medium’—collection of disconnected, *non-interacting* perfectly spherical elastic particles which exchange momentum when they collide with a body
 - ‘Continued Medium’—a continuous chain of particles
- *Several different hypotheses added to the few in Book I*
- ***Includes some small fudges and implausible constructions as well!***



“the greatest production of the human mind.”

Lagrange (1736-1813)

Isaac Newton



English Physicist & Mathematician

25 Dec 1642 – 20 Mar 1727

Newton's Theory of Fluid Resistance (17th Century)

L2

- **Mathematical Principles of Natural Philosophy (July 5, 1687)**

- Book II: Of the Motions of Bodies (in Resisting Mediums)

- **Resistance of bodies moving through a fluid (Proposition 33)**

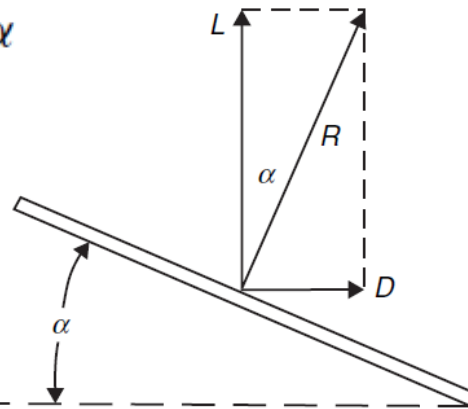
$$D \propto \rho S V^2$$

- *First theoretical derivation of the drag (resistance) force of a body!*

- **Fluid dynamic force on a flat plate**

$$R = \rho V^2 S \sin^2 \alpha$$

$$\frac{L}{D} = \cot \alpha$$



- *Formula is based on Proposition 34...but the formula is not found in Newton's work!*
- *'Rare Medium' fluid model used for this formula*

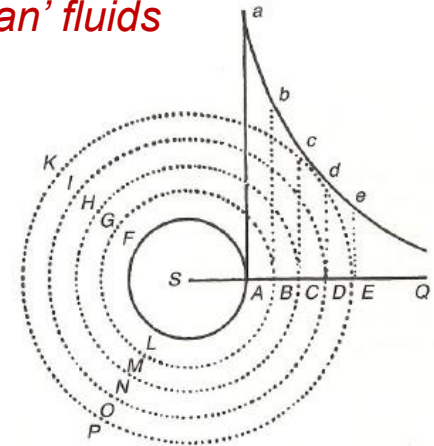
- **Internal resistance within a flow created by its own velocity gradients**

"The resistance arising from the want of lubricity in the parts of a fluid, is, proportional to the velocity with which the parts of the fluid are separated from each other."

- *Provides the well-known linear relationship of shear stress and rate of strain for 'Newtonian' fluids*

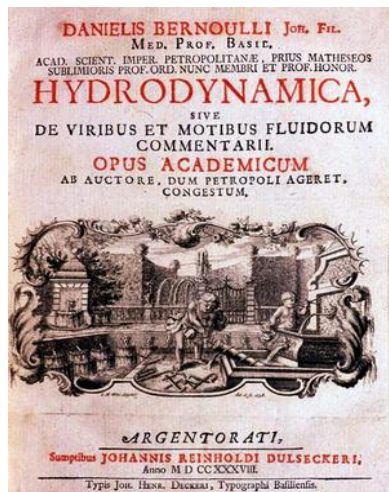
$$\tau \propto dV/dn$$

- *Explains creation of vortex motion about a rotating cylinder in a tank of water*



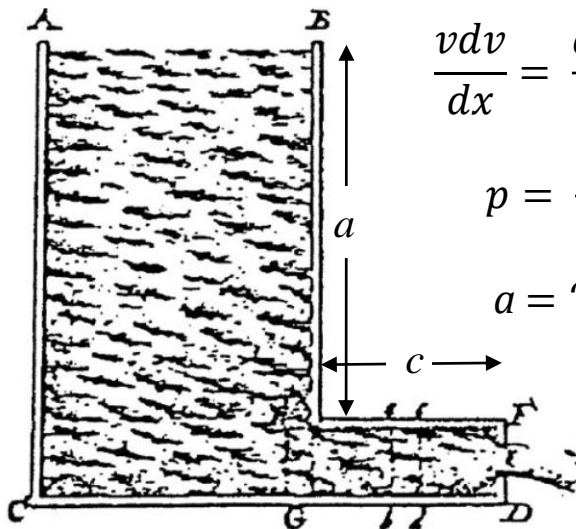
Birth of “Hydrodynamics” (18th Century)

Daniel Bernoulli



1738

Swiss Mathematician
8 Feb 1700 – 17 Mar 1782

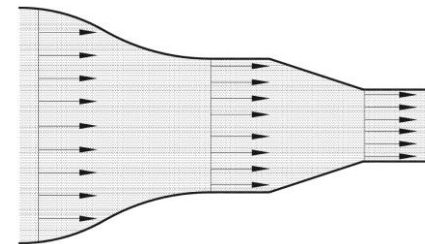


$$\frac{v dv}{dx} = \frac{a - vv}{2c}$$

$$p = \frac{a - v^2}{2c}$$

$a = \text{'head'}$

- Devised parallel-slice hypothesis for flow through ducts
- Analyzed efflux through small opening at the bottom of a vessel that showed compliance with Torricelli's Law
- Daniel Bernoulli successfully derived '**hydraulic-static**' pressure exerted by a moving fluid on the wall of its container--going beyond Stevin's and Pascal's Laws of hydrostatic pressure
- Employed elements of calculus for analysis; used continuity and von Leibniz '*vis viva*' ('live force') or kinetic energy principles; *verified predictions using experiments!*



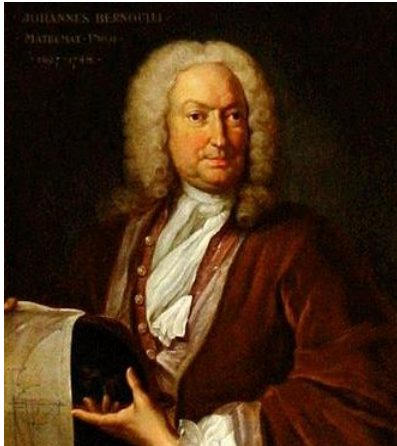
Bernoulli Principle

In a flowing fluid, pressure decreases as velocity increases.

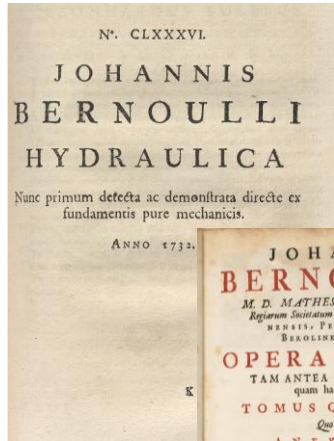
The Well-known Bernoulli's Equation is Not in the Book!

J. Bernoulli's *Hydraulica* (18th Century)

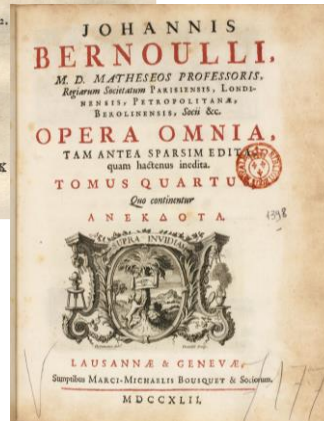
Johann Bernoulli



Swiss Mathematician
6 Aug 1667 – 1 Jan 1748

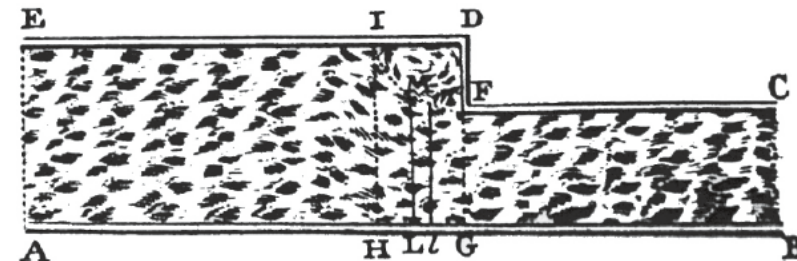


1742



- Analyzed fluid flow through a duct with abrupt change in area using Newton's Laws—instead of 'vis viva' theory used by his son, Daniel Bernoulli

- Inserted whirlpools to convert jump into continuous area variation



- Developed equations of motion of accelerating flow by applying Newton's Second Law to parallel slices of fluid
- Introduced the new concept of convective derivative to account for acceleration due to broadening or narrowing of area—in addition to that due to instantaneous change in velocity
- Generalized Daniel Bernoulli's principle for pressure to non-steady flows

A New Concept of Internal Pressure in Moving Fluids

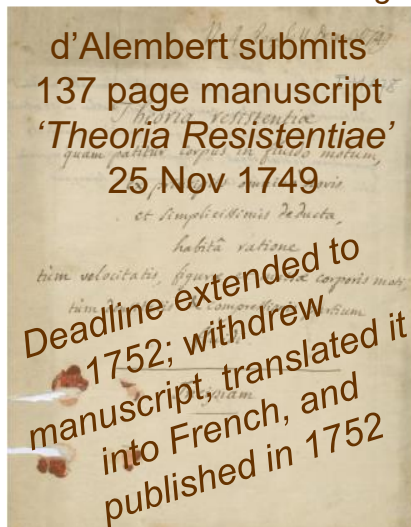
"The force exerted on the sides of a duct while liquid flows through it...is nothing more than the force originating from the compression force by which, certainly, neighboring portions of the fluid are driven one against the other."

Theory of Resistance: A Grand Milestone! ^{L2}

(18th Century)

16 May 1748

Berlin Academy Prize Announced
for Determination of Drag

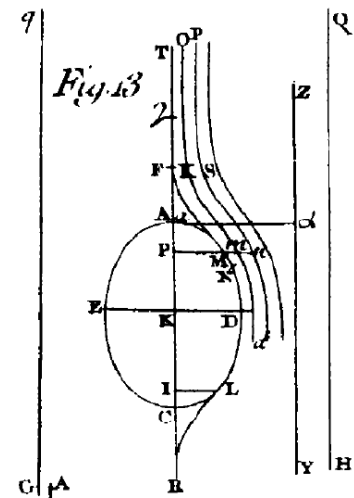


Jean le Rond d'Alembert



French Mathematician
16 Nov 1717 – 29 Oct 1783

- Flow of inviscid fluid about a body is a field of continuous variation in velocity
- Determine the fluid field, then integrate local pressures to find force exerted on the body
- Introduces streamlines, and front and aft stagnation points and zones, for 2-D and axisymmetric bodies
- Develops two equations relating partial derivatives of axial and lateral velocity components to force components for steady flow
- Used his dynamical principle and equilibrium principle to derive hydrodynamical equations for steady, inviscid, incompressible, 2-D and axisymmetric flows
- Uses complex variable transformation and developments in power series in attempts to determine velocity field that is uniform at infinity and tangent to the body along its surface—but unable to solve the equations
- Instead applies his knowledge of Bernoulli's work to estimate drag
- Conclusion: Due to symmetrical fluid field, a symmetrical body "...would suffer no force from the fluid, which is contrary to experience."



"...it seems to me that the theory, developed in all possible rigor, gives, at least in several cases, a strictly vanishing resistance, a singular paradox which I leave to future geometers* to elucidate."

Conclusion Gave Birth to d'Alembert's Paradox

**i.e. mathematicians - the two terms were used interchangeably at that time*

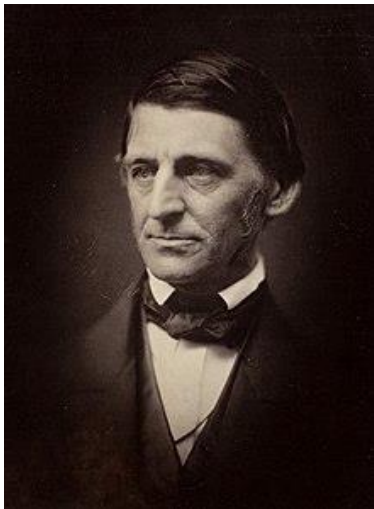
d'Alembert's great strides in the use of mathematics to solve fluid dynamic problems were harbinger of the direction of the field of fluid dynamics for the next 150 years and beyond!

Lecture 2:

Overarching Takeaways

“Everything in Nature Goes by Law, and Not by Luck.”

Ralph Waldo Emerson



25 May 1803 – 27 April 1882

Laws of Nature Serve as Universal Constraints on the Flow of Fluids



Source: AIAA-1982-0315

Lecture 2: Key Takeaways

- **Early Days of Civilization**
 - **Two sets of “Grand Challenge” Problems**
 1. Problems of **Resistance** (ships, water wheels, projectiles,...)
 2. Problems of **Discharge** (water distribution, jet reaction machines,...)
 - **Two Branches of Investigations**
 1. **Hydraulics** (artisan activity based on empirical knowledge)
 2. **Hydrodynamics** (scientific activity based on fundamental laws of nature)
- **Key Foundational Ideas for Fluid Dynamics (*Antiquity to 1750*)**
 - 384-322 BC: Aristotle—*concept of continuum*
 - 287-212 BC: Archimedes—*principles of hydrostatics*
 - 1452-1519: Leonardo da Vinci—*principles of continuity and relative motion*
 - 1586: Stevin—*hydrostatic pressure depends only on the height of the fluid column*
 - 1644: Torricelli—*efflux velocity is proportional to the square root of depth*
 - 1669: Huygens—*resistance is proportional to square of velocity*
 - 1687: Newton—*Laws of Mechanics and theory of fluid resistance*
 - 1738: D. Bernoulli—*pressure decreases as velocity increases*
 - 1742: J. Bernoulli—*concept of internal pressure in moving fluids*
 - 1749: d'Alembert—*symmetrical body would suffer no fluid force--a Paradox!*

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Topic 2

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