Available online at www.sciencedirect.com
ScienceDirect
Computers & Fluids 36 (2007) 1298–1306
computers & fluids
www.elsevier.com/locate/compfluid

Reflection and transmission of plane waves at an interface between two fluids

B. Klenow [a,*], A. Nisewonger [a], R.C. Batra [b], A. Brown [a]

[a] Department of Aerospace and Ocean Engineering, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061, United States
[b] Department of Engineering Science and Mechanics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061, United States

Received 22 July 2006; received in revised form 20 March 2007; accepted 25 March 2007
Available online 22 May 2007

In memory of Professor G.V. Loganathan who was killed on 16 April 2007 during the massacre in Norris Hall, Virginia Tech

Abstract

The propagation, reflection, and transmission of a plane wave through a column of two fluids with a material discontinuity is studied by three methods: a mixed finite element formulation with both pressure and velocity at a point taken as independent variables, and a scaled and an un-scaled acoustic pressure formulation in which only the pressure at a point is taken as an independent variable. It is found that when mass densities of two fluids are close to each other, the un-scaled acoustic pressure formulation gives reasonable results. However, when the speeds of sound in two fluids are close to each other but their mass densities are quite different, and for cases where the first fluid has high impedance relative to that of the second fluid, a mixed or scaled pressure formulation is necessary. Without the mixed or scaled pressure formulation, the continuity conditions at the interface between two fluids are not well satisfied for the un-scaled pressure formulation. The consideration of viscosity of the two fluids and using a dispersion correction method in the time integration scheme in the mixed formulation slightly improves results.

1. Introduction

Understanding wave reflection and transmission at interfaces between two dissimilar materials is of interest in many fields, e.g., composites engineering, geology, and acoustics. Previous work in this area includes that of Mackinnon and Carey [1], and Cordes and Moran [2] who used the standard finite element method (FEM) and Batra, Porfiri, and Spinello [3] who employed a meshless method. These investigators analyzed wave propagation in isotropic and homogeneous linear elastic compressible solids. Other authors have used the finite difference method [4,5] and more recently the FEM [6] to examine acoustic and elastic wave propagation in heterogeneous solid media for geophysics problems. A difference between wave propagation in solids and fluids is that most fluids are taken to be incompressible but solids are generally modeled as compressible. Whereas both distortional and dilatational waves can propagate in a compressible material, only distortional waves propagate in an incompressible material. For a compressible fluid such as air, density changes are considerably more than that in a compressible solid. Thus the problem of wave propagation in two fluids with a material discontinuity may differ significantly from that in two solids. Here we study several formulations of a one-dimensional (1-D) problem involving the reflection and transmission of a plane wave at the boundary between two fluids; such a problem is encountered in acoustics (e.g., see [7]). An analytical solution of the problem is found by transforming the pressure wave equation to the Laplace domain as has been done in [3].

2. Problem formulation and solution

A schematic sketch of the problem studied is shown in Fig. 1. A plane wave propagates in a column of length L

* Corresponding author. Tel.: +1 540 231 9472; fax: +1 540 231 9632.
E-mail address: klenowb@vt.edu (B. Klenow).

0045-7930/$ - see front matter © 2007 Elsevier Ltd. All rights reserved.
doi:10.1016/j.compfluid.2007.03.014

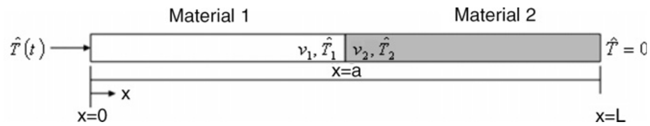

Fig. 1. Schematic sketch of the one-dimensional wave propagation in two distinct fluids.

that is comprised of two distinct fluids, each of length $L/2$, with either the two fluids having nearly equal mass densities and approximately the same speed of sound, or the two fluids having widely different acoustic impedances. We note that the axial velocity, v, and the axial traction, $\widehat{T}$, at the interface between two fluids must be continuous, i.e.,

$$\widehat{T}_1 = \widehat{T}_2 \qquad v_1 = v_2 \tag{1}$$

where subscripts 1 and 2 stand for fluids 1 and 2, respectively.

For an inviscid fluid the axial traction equals the hydrostatic pressure but that is not necessarily true for a viscous fluid.

Initially the two fluids are taken to be at rest and separated by an imaginary membrane of negligible strength and thickness that ruptures as soon as a wave arrives there. This is to keep the two fluids separated which otherwise may mix with each other. Since plane waves are being studied, the continuity of axial traction and velocity at the interface ensures that they do not mix once the fictitious membrane breaks.

2.1. Acoustic pressure formulation

In the acoustic pressure formulation [8,9] it is assumed that disturbances are localized, the fluid is compressible but $\Delta\rho \ll \rho_0$, the reference mass density ρ_0 is a constant, and deformations of the fluid are governed by the Navier–Stokes equations [10,11]:

$$\rho\left(\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j}\right) = \rho b_i + \frac{\partial \sigma_{ij}}{\partial x_j} \tag{2}$$

where

$$\sigma_{ij} = -p\delta_{ij} + \lambda\frac{\partial v_k}{\partial x_k}\delta_{ij} + \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

$\boldsymbol{v}$ is the fluid velocity, p the pressure, $\boldsymbol{b}$ the body force per unit mass, ρ the present mass density, $\Delta\rho$ the change in the mass density, λ and μ fluid viscosities, t the time, $\boldsymbol{x}$ the present position of a material particle, and δ_{ij} the Kronecker delta. These equations are written in rectangular Cartesian coordinates, and a repeated index implies summation over the range of the index. In the acoustic element formulation, it is also assumed that the body force is negligible, the fluid is inviscid, and the convective acceleration is negligible. For the reflection-transmission wave propagation problem, the inviscid assumption is valid because the fluid does not see any effects from a boundary layer, as it would in a fluid–structure interaction problem [12]. With viscosity effects and body forces neglected, the state of stress in the fluid is a hydrostatic pressure [13]. Thus for an one-dimensional problem, Eq. (2) reduces to

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{3}$$

where v is the velocity in the x-direction.

The equation of state for the fluid is taken to be

$$p = c^2\rho \tag{4}$$

where c is the speed of sound in the fluid.

For the one-dimensional problem, the continuity equation is

$$\frac{\partial \rho}{\partial t} = -\rho\frac{\partial v}{\partial x} \tag{5}$$

where the term $\frac{\partial\rho}{\partial x}v$ has been neglected. Eqs. (3)–(5) when combined together give

$$\frac{\partial v}{\partial x} = -\frac{1}{K}\frac{\partial p}{\partial t} \tag{6}$$

$$\frac{\partial^2 p}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0 \tag{7}$$

where $K = \rho c^2$ is the bulk modulus, and we have neglected the term $\frac{\partial\rho}{\partial x}\frac{\partial v}{\partial t}$.

We assume that the fluid is initially at rest, the right end face $x = L$ is traction free, and a time-dependent normal traction given by Eq. (8) with $\overline{P} = 1$ kPa, is applied at the left surface $x = 0$. Fig. 2 shows the variation in the applied pressure.

$$p(0,t) = \overline{P}\sin^2\left(\frac{\pi}{T}t\right)[H(t) - H(t-T)] \tag{8}$$

Here H is the Heaviside step function, T the duration of the pressure pulse, and $\overline{P}$ the amplitude of the pressure pulse. The boundary condition at the traction free surface is inconsequential for our work since numerical solutions have been computed for times prior to the arrival of the wave at the right surface.

2.1.1. Analytical solution

The analytical solution of the problem defined by Eqs. (7) and (8) is obtained by following the procedure outlined

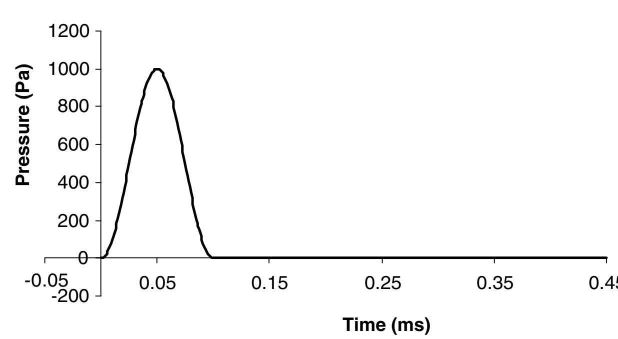

Fig. 2. Variation with time of the pressure applied at the left end face.

in [3]. That is, the problem is transformed to the Laplace domain, the resulting ordinary differential equations are solved, and then the inverse Laplace transform is found. Recalling that a quantity for fluids 1 and 2 is denoted by subscripts 1 and 2 respectively, the Laplace transform of Eq. (7) gives

$$\frac{\partial^2 P_i(x,s)}{\partial x^2} - \frac{s^2}{c_i^2} P_i(x,s) = 0, \quad i = 1, 2, i \text{ not summed} \tag{9}$$

where

$$P_i(x,s) = \int_0^\infty \exp(-st) p_i(x,t) \mathrm{d}t$$

A general solution of Eq. (9) is

$$P_i(x,s) = A_i \exp\left(-\frac{s}{c_i}x\right) + B_i \exp\left(\frac{s}{c_i}x\right) \tag{10}$$

wherein coefficients A_i and B_i are determined from the following boundary and interface conditions in the Laplace transform domain.

$$\begin{aligned} &P_1(0,s) = \widehat{P}(s) \\ &P_1(a,s) = P_2(a,s) \\ &\frac{1}{\rho_1}\frac{\partial P_1}{\partial x}(a,s) = \frac{1}{\rho_2}\frac{\partial P_2}{\partial x}(a,s) \\ &P_2(L,s) = 0 \end{aligned} \tag{11}$$

where

$$\widehat{P}(s) = \int_0^\infty \exp(-st) p(0,t) \mathrm{d}t$$

Once coefficients A_i and B_i are known, pressures in the Laplace domain in the two fluids are given by

$$P_i(x,s) = G_i(x,s)\widehat{P}(s) \tag{12}$$

where $G_1(x,s)$ and $G_2(x,s)$ are the transfer functions. The inverse Laplace transforms of the transfer functions are found by using the shifting theorem in the time domain (see e.g., [14]). The convolution theorem [14] applied to the inverse transforms gives the following expressions for the pressure, $p_i(x,t)$, in the time domain

$$\begin{aligned} p_1(x,t) = &\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{h=0}^{k}\binom{j}{k}\binom{k}{h}(-1)^h \\ &\times\Bigg[\alpha^k\zeta\left(t-2\left(\frac{(j-h)(L-a)}{c_2}+\frac{(j+h-k)}{c_1}a+\frac{x}{2c_1}\right)\right) \\ &-\alpha^{k+1}\zeta\left(t-2\left(\frac{(j-h+1)(L-a)}{c_2}+\frac{(j+h-k)}{c_1}a+\frac{x}{2c_1}\right)\right) \\ &+\alpha^{k+1}\zeta\left(t-2\left(\frac{(j-h)(L-a)}{c_2}+\frac{(j+h-k+1)}{c_1}a-\frac{x}{2c_1}\right)\right) \\ &-\alpha^k\zeta\left(t-2\left(\frac{(j-h+1)(L-a)}{c_2}+\frac{(j+h-k+1)}{c_1}a-\frac{x}{2c_1}\right)\right)\Bigg] \end{aligned}$$

$$\begin{aligned} p_2(x,t) = &\frac{2\rho_2 c_2}{\rho_2 c_2+\rho_1 c_1}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{h=0}^{k}\binom{j}{k}\binom{k}{h}\alpha^k(-1)^h \\ &\times\Bigg[\zeta\left(t-2\left(\frac{(j-h)(L-a)}{c_2}+\frac{x-a}{2c_2}+\frac{(j+h-k)}{c_1}a+\frac{a}{2c_1}\right)\right) \\ &-\zeta\left(t-2\left(\frac{(j-h)(L-a)}{c_2}+\frac{2L-x-a}{2c_2}+\frac{(j+h-k)}{c_1}a+\frac{a}{2c_1}\right)\right)\Bigg] \end{aligned}$$

where

$$\begin{aligned} &\alpha = \frac{\rho_2 c_2 - \rho_1 c_1}{\rho_2 c_2 + \rho_1 c_1} \\ &\zeta(\beta) = \begin{cases} P\sin^2\left(\frac{\pi}{T}\beta\right)[H(\beta)-H(\beta-T)] & \text{if } 0 \leqslant \beta < t \\ 0 & \text{if } \beta < 0 \text{ or } \beta \geqslant t \end{cases} \end{aligned} \tag{13}$$

and $p_1(x,t)$ holds for $x \in (0,a)$, and $p_2(x,t)$ holds for $x \in (a,L)$.

For fixed values of x and t, the upper limit for j in the above two infinite series is taken to be 100.

2.1.2. Numerical solution of the problem

2.1.2.1. Scaled and un-scaled pressure formulations. An approximate solution of the 1-D wave problem formulated above is found by using the FEM with the region [0, L] divided into uniform elements and one node placed at the interface between the two fluids. For the inviscid fluids the continuity conditions (1) at the interface between them can be taken as the continuity of pressure and the continuity of volume flow [15]. That is

$$\begin{aligned} &p|_{x=a^-} = p|_{x=a^+} \\ &\frac{1}{\rho_1}\frac{\partial p}{\partial x}\bigg|_{x=a^-} = \frac{1}{\rho_2}\frac{\partial p}{\partial x}\bigg|_{x=a^+} \end{aligned} \tag{14}$$

By placing a node at the interface, the continuity of pressure is automatically enforced but volume flow continuity is not necessarily satisfied. It is known that for problems containing a material discontinuity the standard weak formulation of the afore-stated problem does not properly enforce the volume flow continuity over the interface. To enforce it we first multiply both sides of Eq. (7) with ϕ/ρ, where ϕ is a test function, and then integrate the resulting equation over the region [0, L], thereby obtaining Eq. (15).

$$\int_0^L \left[\frac{\partial^2 p}{\partial x^2}\frac{\phi}{\rho} - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}\frac{\phi}{\rho}\right]\mathrm{d}x = 0 \tag{15}$$

This procedure is analogous to that used by Christiansen and Krenk [15]. Integrating the first term in Eq. (15) by parts, yields a weak formulation of the problem, given below as Eq. (16), where the continuity of the volume flow at the interface of the two fluids has been enforced and ρ has been assumed to be constant in each fluid.

$$\int_0^{L^-/2} \frac{1}{\rho}\frac{\partial p}{\partial x}\frac{\partial \phi}{\partial x}\mathrm{d}x + \int_{L^+/2}^{L} \frac{1}{\rho}\frac{\partial p}{\partial x}\frac{\partial \phi}{\partial x}\mathrm{d}x + \int_0^{L^-/2} \frac{1}{\rho c^2}\frac{\partial^2 p}{\partial t^2}\phi \mathrm{d}x + \int_{L^+/2}^{L} \frac{1}{\rho c^2}\frac{\partial^2 p}{\partial t^2}\phi \mathrm{d}x = \frac{1}{\rho}\frac{\partial p}{\partial x}\bigg|_L - \frac{1}{\rho}\frac{\partial p}{\partial x}\bigg|_0 \quad (16)$$

The pressure field p and the test function ϕ are approximated by using piecewise linear basis functions. The mass matrix is lumped using the row-sum technique. The resulting system of coupled ordinary differential equations is integrated by the unconditionally stable and second-order accurate Newmark family of methods with $\gamma = 1/2$ and $\beta = 1/4$. For $L = 0.4$ m, numerical experiments with different meshes, time step sizes, and four combinations of fluids, revealed that the computed solution converged for a uniform mesh of 160 elements and a time step equal to the time taken for the wave to propagate through 1/2 of the length of an element in the first fluid.

Values of material parameters for the fluids studied herein are listed in Table 1, and the four combinations of fluids used in the numerical solution of the problems are listed in Table 2. Water and oil represent the similar density case (Problem 1), water and mercury the similar sound speed case (Problem 2), and water and air the high impendence to low impedance case (Problem 3). Results are presented as the non-dimensional pressure versus the non-dimensional x-coordinate at a given time. Pressure is non-dimensionalized by the maximum of the applied pressure, which in this problem is 1 kPa.

For each case we also include a FE solution obtained using the un-scaled pressure formulation in which the test function in Eq. (15) is ϕ rather than ϕ/ρ. In the un-scaled pressure formulation the continuity condition $(14)_2$ is not necessarily satisfied unless one uses a Lagrange multiplier or a penalty parameter which is not done here. The comparison of results for the scaled and the un-scaled pressure formulations will delineate the advantages of properly scaling the pressure field and automatically satisfying the continuity condition $(14)_2$ at the interface between the two fluids.

For Case 1 (water–oil), Fig. 3 compares the analytically and the numerically computed spatial variations of the pressure at $t = 257$ μs. It takes 133.4 μs for the wave to arrive at the interface and at $t = 298.6$ μs it arrives at the right end face. Thus the spatial variation of the pressure depicted in Fig. 3 is after it has been reflected from and transmitted through the interface, but before the wave transmitted through fluid 2 arrives at the right end. It is transparent that the wave profile has been noticeably altered because of reflections and transmissions through the interface, and solutions computed with the scaled and the un-scaled pressure formulations are close to the analytical solution of the problem. The numerical solution with the scaled pressure formulation essentially coincides with the analytical solution of the problem.

The spatial variations of the pressure for Case 2 (water–mercury) at $t = 257$ μs are shown in Fig. 4. The plane wave takes approximately 138 μs to propagate through 200 mm of mercury column. As expected, results plotted in Fig. 4 show that the FE solution for the un-scaled pressure formulation differs significantly from the analytical solution of the problem, as it does not capture the reflected wave well and erroneously predicts that the incident wave in water is entirely transmitted into the mercury. The scaled

Table 1
Values of density (ρ), bulk modulus (K), speed of sound (c), and acoustic impedance (Z) for the fluids studied

Fluid	ρ (kg/m^3)	K (GPa)	c (m/s)	Z (MPa · s/m)
Water	1025	2.3	1498	1.53
Oil	920	1.35	1211	1.11
Mercury	13595	28.5	1448	19.6
Air	1.25	148×10^{-6}	344	4.3×10^{-4}
Hypothetical	1025	2560×10^{-6}	500	0.513

Table 2
Combinations of fluids with impedance ratios for the four problems studied

Problem no.	1st fluid	2nd fluid	Z_1/Z_2
1	Water	Oil	1.38
2	Water	Mercury	0.078
3	Water	Air	3558
4	Water	Hypothetical	2.92

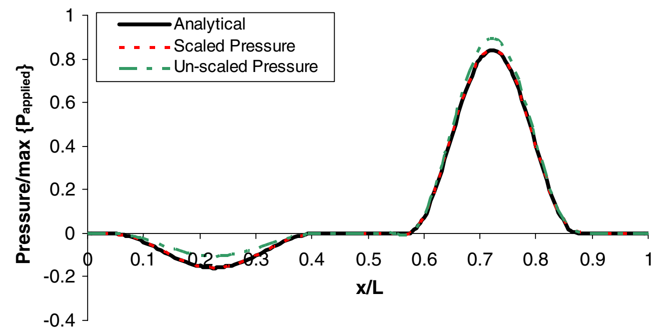

Fig. 3. Spatial variation of pressure at $t = 0.257$ ms for plane wave propagation in water–oil using the acoustic pressure formulations.

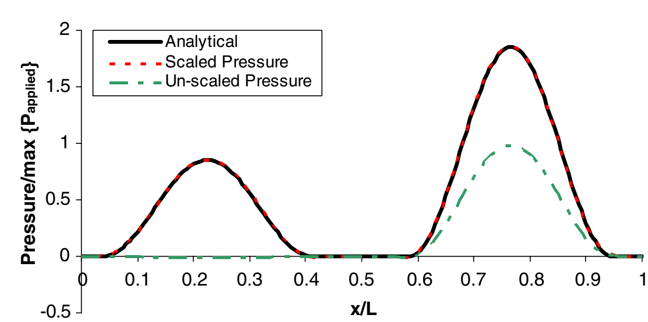

Fig. 4. Spatial variation of pressure at $t = 0.257$ ms for plane wave propagation in water–mercury using the acoustic pressure formulations.

pressure formulation results do not exhibit this unphysical behavior and agree well with the analytical solution. The difference between the un-scaled and the scaled pressure formulations is that the former does not impose the continuity of volume flow at the interface. This was not an issue in Case 1 where the two fluids have nearly the same mass densities, because then the densities in the continuity Eq. $(16)_2$ cancel out. However, as results for Case 2 show, this is not true when densities of the two fluids are significantly different and enforcing volume flow continuity condition at the interface becomes critical for computing accurate results.

In order to illustrate this further, we study wave propagation in a column of water and a hypothetical fluid, given in Table 2 as Case 4, that has the same mass density as water but sound speed of 500 m/s; these results are shown in Fig. 5. The results for both pressure formulations agree well with those obtained from the analytical solution which gives the non-dimensional amplitudes of the reflected and the transmitted waves to be −0.49 and 0.5, respectively.

We now study the performance of the scaled and the un-scaled pressure formulations for plane wave propagation through a column of water and air that have drastically different values of the mass density and the speed of sound. We expect the incident wave to be almost entirely reflected back into water, which is consistent with the laws of wave reflection and transmission from a high acoustic impedance medium into a low acoustic impedance medium [7]. As shown in Fig. 6, the scaled pressure formulation gives expected pressure variation while the un-scaled pressure formulation yields a response similar to that of Case 1, i.e., reflection and transmission. We thus conclude that only the scaled pressure formulation gives good results for wave propagation in a column of two fluids having widely different values of mass densities and speeds of sound. Furthermore, we find that the scaled pressure formulation also gives good results for the case where the two fluids have widely different mass densities but the same speeds of sound.

2.2. Mixed formulation

Another approach to studying the acoustic reflection and transmission problem is to use a mixed formulation [4–6] in which both the axial velocity and the pressure are taken as unknowns at each node, and Eqs. (3) and (6) are simultaneously solved. By placing a node at the interface between two fluids the two interface continuity conditions (1) are automatically satisfied. The number of unknowns in the mixed formulation equals twice of that in the scaled pressure formulation.

The Galerkin formulation of the problem is derived by using piecewise linear basis functions, domain integrals

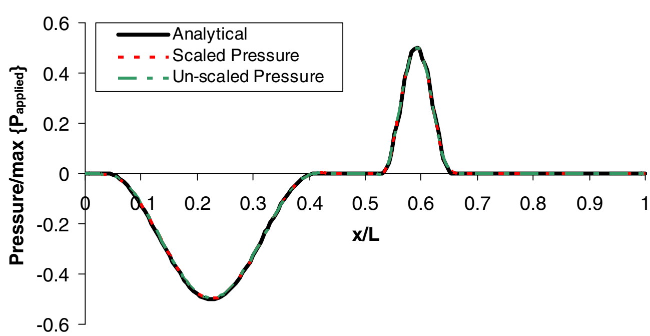

Fig. 5. Spatial variation of pressure at $t = 0.257$ ms for plane wave propagation in water–hypothetical fluid using the acoustic pressure formulations; the three solutions are virtually coincident.

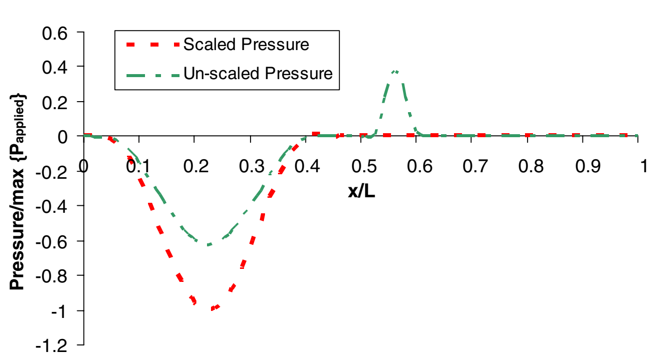

Fig. 6. Spatial variation of pressure at $t = 0.257$ ms for plane wave propagation in water–air using the acoustic pressure formulation.

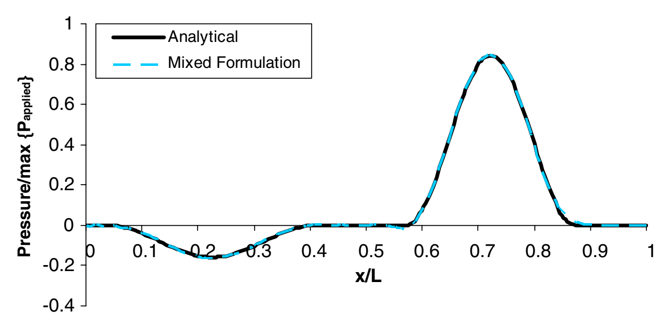

Fig. 7. Spatial variation of pressure at $t = 0.257$ ms for plane wave propagation in water–oil using the mixed formulation; the two solutions essentially coincide with each other.

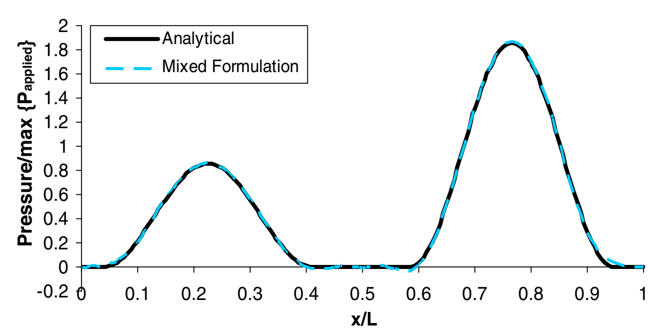

Fig. 8. Spatial variation of pressure at $t = 0.257$ ms for plane wave propagation in water–mercury using the mixed formulation; the two solutions essentially coincide with each other.

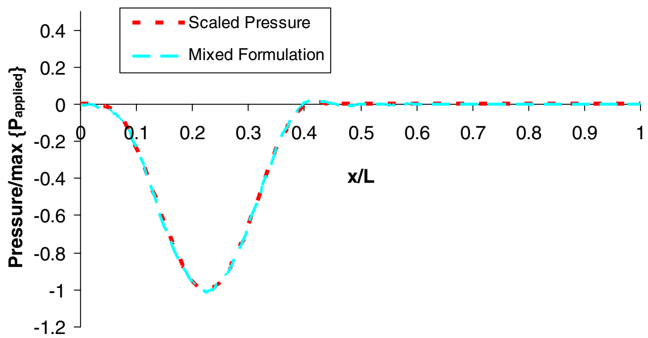

Fig. 9. Spatial variation of pressure at $t = 0.257$ ms for plane wave propagation in water–air using the mixed formulation; the two solutions are virtually coincident.

are evaluated exactly, the lumped mass matrix is obtained with the row-sum technique, and the resulting coupled ordinary differential equations are integrated with the unconditionally stable Crank–Nicolson method. Numerical results are computed with a mesh of 160 uniform elements with one node at the interface and a time step equal to the time taken for the wave to propagate through 1/2 of the length of an element in the first fluid.

Computed spatial variations of the pressure for the first three cases studied in Section 2.1 are depicted in Figs. 7–9. The mixed formulation gives very good results in all three cases with the numerical solution agreeing well with the corresponding analytical solution; thus it can be used to solve acoustic reflection and transmission problems.

3. Comparison of results from the pressure and the mixed formulations

The L^2 error norms for the results from the water–oil and water–mercury combinations are listed in Table 3. For all cases studied the scaled pressure formulation yields better results than the mixed formulation, but both give significantly superior results than the un-scaled pressure formulation.

Table 3
L^2 error norms over the reflected wave (fluid 1 in Fig. 1), transmitted wave (fluid 2 in Fig. 1), and the entire fluid domain for two reflection–transmission problems

	Entire domain	Reflected wave	Transmitted wave
Water–oil (mixed formulation)	0.027	0.044	0.026
Water–oil (scaled pressure formulation)	0.009	0.008	0.009
Water–oil (un-scaled pressure formulation)	0.093	0.335	0.063
Water–mercury (mixed formulation)	0.017	0.018	0.017
Water–mercury (scaled pressure formulation)	0.008	0.008	0.008
Water–mercury (un-scaled pressure formulation)	0.607	1.020	0.470

4. Effect of mass matrix and the time step size

4.1. Consistent vs. lumped mass matrices

Results presented in previous sections were computed by using lumped mass matrices because the use of the lumped mass matrix is computationally less expensive than that of the consistent mass matrix. We now compare results obtained using the lumped and the consistent mass matrices for both the scaled pressure formulation and the mixed formulation.

Fig. 10 exhibits results computed with the consistent and the lumped mass matrices for both the scaled pressure and the mixed formulation using 40 uniform elements in the domain. In each case, results with the consistent mass matrix are closer to the analytical solution of the problem than those with the lumped mass matrix. However, with 100 uniform elements in the scaled pressure formulation, results with the two mass matrices are very close to each other as can be seen from the L^2 error norms listed in Table 4. With 100 uniform elements in the mixed formulation, results with the two mass matrices are also close to each other, although not as close as in the scaled pressure formulation. With both 40 and 100 elements and the lumped mass matrix, the scaled pressure formulation gives better results than the mixed formulation. We note that the numerical solution for both formulations converges when 160 elements are used, and the two sets of results are virtually identical to each other.

4.2. Time step size

Results for both the scaled pressure and the mixed formulation were obtained using unconditionally stable implicit time integration schemes. In Section 2.1.2, we noted that the time step Δt used to obtain results was such that in one time step the wave propagated through 1/2 of the length of an element in the first fluid. When using the Newmark method with $\gamma = 1/2$ and $\beta = 1/4$ for the scaled pressure formulation, we found that a time step as large as $5\Delta t$ gave accurate results. However, this is not true when using the

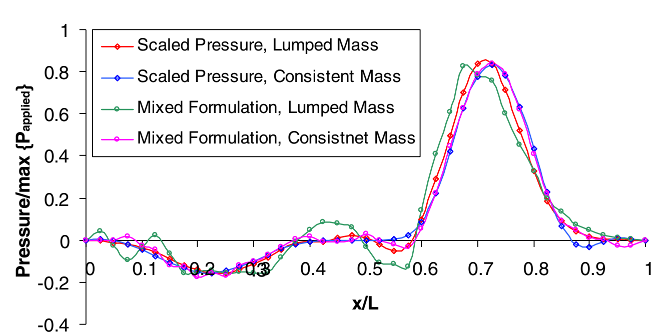

Fig. 10. Spatial variations of pressure at $t = 0.257$ ms with 40 elements for plane wave propagation in water–oil using the mixed formulation and the scaled pressure formulation with both lumped and consistent mass matrices.

Table 4
L^2 error norms over the reflected wave (fluid 1 in Fig. 1), transmitted wave (fluid 2 in Fig. 1), and the entire fluid domain for the water–oil combination with lumped and consistent mass matrices

Formulation	Water–oil combination	Entire domain	Reflected wave	Transmitted wave
Mixed	Consistent 40 elements	0.065	0.168	0.056
	Lumped 40 elements	0.283	0.583	0.262
	Consistent 100 elements	0.012	0.013	0.012
	Lumped 100 elements	0.047	0.057	0.046
Scaled pressure	Consistent 40 elements	0.047	0.038	0.047
	Lumped 40 elements	0.106	0.095	0.106
	Consistent 100 elements	0.014	0.012	0.014
	Lumped 100 elements	0.016	0.015	0.016

Crank–Nicolson method for the mixed formulation. In this case the time step must be taken such that the wave propagates at most through one element per time step. This is because for the Crank–Nicolson method and the mixed formulation, the amplification factor, A, is given by

$$A = \frac{h - c\Delta t}{h + c\Delta t} \tag{17}$$

where h is the element length. Therefore if the wave propagates through more than one element in a time step, A falls into the range $-1 < A < 0$, and the solution oscillates. We have chosen to reduce the time step to avoid oscillations. This makes the mixed formulation computationally more expensive than the scaled pressure formulation. However, if larger time steps are desired in the mixed formulation, one could include an appropriate artificial damping term to either eliminate or reduce oscillations.

5. Modifications to the mixed formulation

5.1. Viscous fluids

Moura [16] and Ludwig and Levin [17] have shown that considering viscosity in an acoustic wave propagation problem can be important. In order to delineate this, we modify the mixed formulation to include viscosity by using Stokes's approximation [10], i.e., $\lambda = -2/3\mu$, or equivalently setting the bulk viscosity equal to zero. Thus assuming that the mass density of each fluid does not vary much in the axial direction, equations governing 1-D deformations of the viscous fluid become

$$\frac{\partial v}{\partial x} = -\frac{1}{K}\frac{\partial p}{\partial t} \tag{18}$$

$$\rho\frac{\partial v}{\partial t} = -\frac{\partial p}{\partial x} + \frac{4}{3}\mu\frac{\partial^2 v}{\partial x^2} \tag{19}$$

When viscosity is considered, the traction continuity condition $(1)_2$ at the interface involves both the pressure and μ $(\partial v/\partial x)$.

To test the effect of viscosity in the reflection–transmission problem, wave propagation in a column of water–oil was analyzed with $\mu_1 = 8.9 \times 10^{-4}$ Pa s and $\mu_2 = 1$ Pa s. No discernable difference in results occurred with the consideration of viscosity. However, the effect of viscosity is not perceptible in this case because the length of the fluid column is only 0.4 m. In order to see the effect of viscosity more clearly, we first artificially increased the viscosity of oil to 1500 Pa s; the corresponding results are depicted in Figs. 11 and 12 which vividly reveal that the transmitted wave begins to disperse as it travels through the highly viscous oil. Subsequently, we increased the length of the fluid column to 1 m but kept the viscosity of oil as 1 Pa s. Results computed with 200 uniform elements and a time step equal to that needed for the wave to propagate through 1/2 of the length of an element in the first fluid, depicted in Fig. 13, show that over a large distance the viscosity of both the water and the oil dissipates the propagating waves (incident, transmitted and reflected). Thus when studying wave reflection–transmission over large lengths the mixed formulation of the problem that includes viscosity should be used.

5.2. Dispersion correction

We have used implicit time integration schemes which when used with consistent mass matrices generally provide more accurate results than explicit methods [18] employing lumped mass matrices. However, implicit methods are computationally more expensive because a system of simultaneous equations must be solved at each time step.

The accuracy of implicit schemes is important in wave propagation problems because numerical integration techniques tend to disperse waves; this effect is more noticeable with explicit methods. However, explicit techniques are

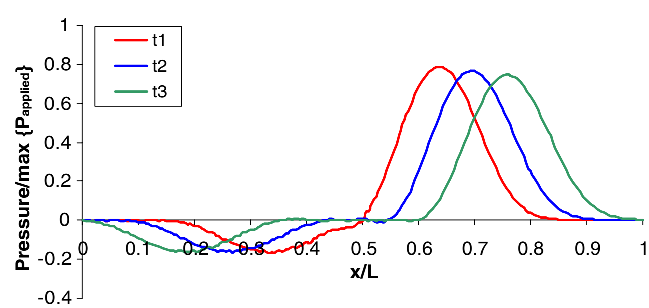

Fig. 11. Spatial distribution of pressure with artificially increased viscosity of oil at 0.23 ms (t1), 0.25 ms (t2), and 0.27 ms (t3).

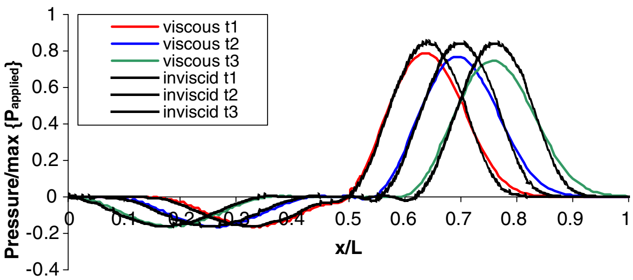

Fig. 12. Comparison of the spatial distribution of pressure with and without the consideration of artificially increased viscosity of oil at 0.23 ms (t1), 0.25 ms (t2), and 0.27 ms (t3).

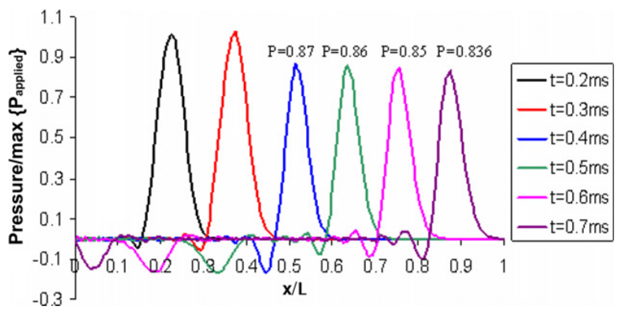

Fig. 13. Spatial distribution of pressure in the 1 m water–oil column at six times.

desirable because they are computationally less expensive than implicit schemes. A less computationally expensive scheme is especially well suited for the mixed formulation because of the time step limits discussed in Section 2.2. Krenk [18] has proposed a dispersion correction to modify explicit integration schemes so that they behave like implicit schemes; we use it to study the present problem. It involves modifying the mass matrix as follows:

$$M_{\mathrm{d}} = (1+\bar{\gamma})M_{\mathrm{lumped}} - \bar{\gamma}M_{\mathrm{consistent}} \tag{20}$$

where $\bar{\gamma}$ is a constant.

The motivation for this modification is that in most problems the ideal mass matrix for accuracy is between a fully lumped and a fully consistent matrix [18]. The advantage of this method is that in the integration scheme only a lumped mass matrix is inverted, thus the efficiency of the explicit method is retained while the dispersion matrix, M_{d}, partially restores the accuracy of the implicit scheme. Krenk gives optimal value of the constant $\bar{\gamma}$ as 1/2 but we experimented with both 1/2 and 3/4.

Here we use the explicit scheme as the standard trapezoidal method [19], and implement the dispersion correction as follows:

$$\begin{aligned} &(M_{\mathrm{lumped}} + \alpha\Delta t K)v_{n+1} = F_{n+1} - K\tilde{d}_{n+1} \\ &\tilde{d}_{n+1} = d_n + (1-\alpha)M_{\mathrm{d}}M_{\mathrm{lumped}}^{-1}\Delta t v_n \\ &d_{n+1} = \tilde{d}_{n+1} + \alpha M_{\mathrm{d}}M_{\mathrm{lumped}}^{-1}\Delta t v_{n+1} \end{aligned} \tag{21}$$

where v_{n+1} denotes the value of v at time t_{n+1}, and $0 \leqslant \alpha \leqslant 1$. The column matrix $\boldsymbol{v}$ is comprised of values at nodes of the first time derivative of axial velocity and pressure. The column matrix $\boldsymbol{d}$ is comprised of values at nodes of axial velocity and pressure. To see the effectiveness of the dispersion correction method, we study a "worst case" scenario by setting $\alpha = 0$ in Eq. (21), taking 40 uniform elements over the entire fluid column, and time step Δt^* equal to the time taken for the wave to propagate through 2/100 of the length of an element in the first fluid. Fig. 14 depicts the solution for wave propagation in the water–oil column using a lumped mass matrix (explicit), a consistent mass matrix (implicit), and the dispersion correction matrix (implicit–explicit) with two values of $\bar{\gamma}$; the L^2 error norms are listed in Table 5. These results show that with the dispersion correction term included, the wave shape in both the water and the oil is slightly better preserved than that in the fully explicit method. The dispersion correction also damps out numerical oscillations as the wave propagates through the water–oil interface. Taking $\bar{\gamma} = 3/4$ tends to damp out these numerical oscillations more effectively. Results plotted in Fig. 15 and the L^2 error norms given in Table 6 show that the implicit–explicit results of Fig. 14 can be improved by increasing the total number of elements from 40 to 60. Thus incorporating the dispersion correction term in the mixed formulation is beneficial as it improves the accuracy of the explicit method. This saves computational time as smaller number of elements can be used, which is desirable due to the small time step requirement of the mixed formulation. While wave dispersion and computational efficiency are not a major concern in the present problem due to the small

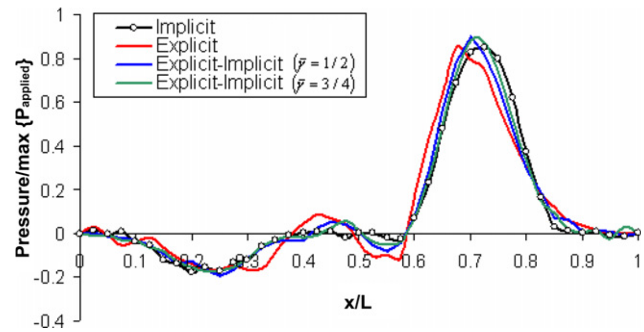

Fig. 14. Comparison of the spatial variation of pressure at $t = 0.257$ ms computed with and without the dispersion correction for a mesh of 40 uniform elements.

Table 5
L^2 error norms over the reflected wave (fluid 1 in Fig. 1), transmitted wave (fluid 2 in Fig. 1), and the entire fluid domain for the 40 element water–oil combination results with and without dispersion correction

	Entire fluid	Reflected wave	Transmitted wave
Explicit–implicit, $\bar{\gamma} = 1/2$	0.183	0.275	0.179
Explicit–implicit, $\bar{\gamma} = 3/4$	0.117	0.232	0.115
Explicit	0.284	0.556	0.270
Implicit	0.056	0.167	0.048

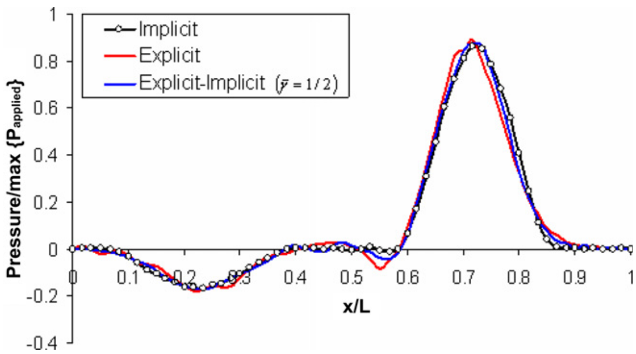

Fig. 15. Comparison of the spatial variation of pressure at $t = 0.257$ ms computed with and without the dispersion correction for a mesh of 60 uniform elements.

Table 6
L^2 error norms over the reflected wave (fluid 1 in Fig. 1), transmitted wave (fluid 2 in Fig. 1), and the entire fluid domain for the 60 element water–oil combination results with and without dispersion correction

	Entire fluid	Reflected wave	Transmitted wave
Explicit–implicit $\bar{\gamma} = 1/2$	0.064	0.121	0.059
Explicit	0.112	0.219	0.104
Implicit	0.045	0.068	0.043

domain, they ought to be considered in problems involving larger domains.

6. Conclusions

To properly capture reflection and transmission of a plane wave through a column of fluid with material discontinuities, the continuity of tractions and velocity (or the volume flow) at the interface between two fluids must be well satisfied. When mass densities of the two fluids are close to each other, the un-scaled acoustic pressure formulation in which the continuity of volume flow is not necessarily well satisfied, gives good results. However, when either the speeds of sound in the two fluids are very close to each other but their mass densities are quite different, or the acoustic impedances of the two fluids are quite different, then the scaled pressure formulation or a mixed formulation in which the velocity and the pressure at a point are taken as independent variables should be employed. The scaled pressure formulation gives better results than the mixed formulation, although both methods perform exceptionally well when compared to the un-scaled pressure formulation. The scaled pressure formulation is potentially less computationally expensive than the mixed formulation because a larger time step can be used and the number of unknowns in the former is one-half of that in the latter. However, the mixed formulation can be advantageous for use in fluid–structure reflection and transmission problems. The computational algorithm for the mixed formulation can be improved by incorporating the dispersion correction and the fluid viscosity.

Acknowledgements

We are indebted to Dr. Davide Spinello for his help in deriving the analytical solution of the problem. RCB's work was partially supported by the ONR grant N00014-98-06-0567 to Virginia Polytechnic Institute and State University with Dr. Y. D. S. Rajapakse as the program manager. Views expressed herein are those of authors and are neither of the funding agency nor of VPI&SU.

References

[1] Mackinnon RJ, Carey GF. Treatment of material discontinuities in finite element computations. Int J Numer Meth Eng 1987;24:393–417.
[2] Cordes LW, Moran B. Treatment of material discontinuity in the element-free Galerikin method. Comput Meth Appl Mech Eng 1996;139:75–89.
[3] Batra RC, Porfiri M, Spinello D. Free and forced vibrations of a segmented bar by a meshless local Petrov-Galerkin (MLPG) formulation. Comput Mech 2006;DOI 10.1007/S00466-006-0049-6 (available online).
[4] Virieux J. P-SV wave propagation in heterogeneous media: Velocity–stress finite-difference method. Geophysics 1986;51:889–901.
[5] Sochacki JS, George JH, Ewing RE, Smithson SB. Interface conditions for acoustic and elastic wave propagation. Geophysics 1991;56:168–81.
[6] Zhang J, Verschuur DJ. Elastic wave propagation in heterogeneous anisotropic media using the lumped finite-element method. Geophysics 2002;67:625–38.
[7] Skudrzyk E. The foundations of acoustics. New York: Springer-Verlag; 1971.
[8] Zienkiewicz OC, Bettess P. Fluid–structure dynamic interaction and wave forces: an introduction to numerical treatment. Int J Numer Meth Eng 1978;13:1–16.
[9] Muller CW. Simplified analysis of linear fluid–structure interaction. Int J Numer Meth Eng 1981;17:113–21.
[10] Tsai SJS. Power transformer partial discharge (PD) acoustic signal detection using fiber sensors and wavelet analysis, modeling, and simulation. Master's Thesis, Dept. of Electric and Comp. Eng., Virginia Tech, 2002.
[11] Felippa CA, DeRuntz JA. Acoustic fluid volume modeling by the displacement potential formulation, with emphasis on the wedge element. Comp Struct 1991;41:669–86.
[12] Lamb H. Hydrodynamics. 6th ed. Cambridge: Cambridge University Press; 1932.
[13] Karamcheti K. Principles of ideal-fluid aerodynamics. New York: Wiley; 1980.
[14] Greenberg M. Advanced engineering mathematics. 2nd ed. New Jersey: Prentice-Hall; 1998.
[15] Christiansen PS, Krenk S. A recursive finite element technique for acoustic fields in pipes with absorption. J Sound Vib 1988;122:107–18.
[16] Moura A. Causal analysis of transient viscoelastic wave propagation. J Acoust Soc Am 2006;119:751–5.
[17] Ludwig R, Levin PL. Analytical and numerical treatment of pulsed wave propagation into a viscous fluid. IEEE Trans Ultrason Ferr 1995;42:789–92.
[18] Krenk S. Dispersion-corrected explicit integration of the wave equation. Comput Meth Appl Mech Eng 2001;191:975–87.
[19] Hughes TJR. The finite element method: linear static and dynamic finite element analysis. Mineola: Dover; 2000.