Global Asymptotic Stabilization of an Underwater Vehicle Using Internal Rotors [1]

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Abstract

We derive a feedback control law that uses internal rotors to globally stabilize steady translation of an underwater vehicle that is subject to fluid drag as well as conservative rigid body and fluid forces. This result extends our previous work on asymptotic stabilization of the system where physical dissipation was neglected. Our stabilizing control laws consist of two terms: the first term addresses the conservative part of the system by shaping energy to yield Lyapunov stability, and the second term adds dissipation to ensure asymptotic stability to the motion of interest. The Lyapunov function must be suitably modified in the second step since fluid drag destroys conservation of momentum, and the feedback-controlled dissipation must be carefully designed to dominate the possibly destabilizing fluid drag.

1 Introduction

Underwater vehicles designed to swim efficiently are often stabilized and controlled by movable tail fins. Fins offer stability and reasonable control authority at moderate vehicle speeds, but they lose effectiveness at low speeds. Internal rotors can provide actuation, even at low speeds, by means of angular momentum exchange. Internal actuators are attractive because they are protected from the corrosive seawater environment and they do not directly contribute to vehicle drag. Internal rotors may thus complement conventional means of underwater vehicle actuation by extending the operating regime, improving reliability, and providing robustness to actuator failure.

In this paper we continue work begun in [10, 11] on stabilization of underwater vehicle dynamics using internal rotors. Our main contribution here is to provide global asymptotic stabilization in the presence of fluid drag which was neglected in the model of [10, 11].

Our approach consists of two steps. In the first step we ignore the fluid drag and treat the vehicle dynamics as a Hamiltonian system. We define a control law that stabilizes a desired steady motion while preserving the Hamiltonian structure. The control shapes the kinetic energy (Hamiltonian) of the closed-loop system to make an otherwise unstable equilibrium motion Lyapunov stable (see [1, 2, 3] for further background and generalizations of this step). The Lyapunov function for the closed-loop system is generated by the energy-Casimir method and the equilibrium is a maximum.

In the second step, we add feedback-controlled dissipation to ensure asymptotic stability to the motion of interest. In [11] the Lyapunov function computed in the first step is used to determine the dissipative control term. Dissipation is added to increase the system energy to the equilibrium value. Because the rotors are internal, this dissipation does not destroy conservation of momentum.

In this paper, however, the choice of feedback-controlled dissipation must provide stabilization in the presence of the fluid drag which destroys the momentum conservation laws. This has the effect of making indefinite the rate of change of the Lyapunov function derived in the first step. Further, since the equilibrium is a maximum for the conservative part of the dynamics, the fluid drag, which decreases energy, can be destabilizing if it is not properly dominated by the feedback-controlled dissipation.

In Section 2, we present the dynamic model for an underwater vehicle with internal rotors subject to viscous forces and moments. In Section 3, we recall the control law of [11] which stabilizes long axis translation for the conservative system. In Section 4, we add feedback dissipation that overcomes the destabilizing effect of the

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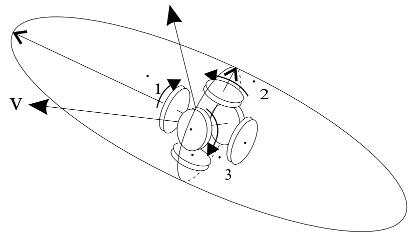

Figure 2.1: Vehicle with three internal rotors.

fluid drag and ensures global asymptotic stability to the desired motion. This requires a modification of the previously developed Lyapunov function to a form which is negative semidefinite and a careful argument based on LaSalle's invariance principle.

2 Dynamic Model

We consider a low-dimensional vehicle model based on Kirchhoff's equations. These equations describe a six degree-of-freedom, neutrally buoyant rigid body in an infinite volume of ideal, irrotational fluid that is at rest infinitely far from the body [8]. Viscous effects and external control inputs are then added to this conservative rigid body model as external forcing.

The vehicle is modeled as an ellipsoidal body of mass m. Let the matrix $\boldsymbol{I}$ be the sum of the body inertia and the *added inertia* from the potential flow model of the fluid. Similarly, $\boldsymbol{M}$ denotes the sum of the body mass m multiplied by the identity matrix and the *added mass matrix*. We assume that m is also the mass of the displaced fluid so that the vehicle is neutrally buoyant. For an ellipsoid with uniformly distributed mass, $\boldsymbol{I}$ and $\boldsymbol{M}$ are diagonal in a coordinate system defined by the ellipsoid principal axes. In these body coordinates, the vehicle moves through the fluid with translational velocity $\boldsymbol{v} = [v_1, v_2, v_3]^T$ and angular velocity $\boldsymbol{\Omega} = [\Omega_1, \Omega_2, \Omega_3]^T$.

The length of the ith principal axis of the ellipsoid is denoted L_i and we assume that $L_1 > L_2 > L_3$. Let the diagonal elements of $\boldsymbol{M}$ be (m_1, m_2, m_3) and the diagonal elements of $\boldsymbol{I}$ be (I_1, I_2, I_3). For the given axis length ordering, it is always true that $m_3 > m_2 > m_1$. The ordering of the inertia elements depends on the relative lengths of the semiaxes [9, 6].

We model the actuators as three symmetric rotors whose spin axes are aligned with the ellipsoid principal axes. We also assume that the rotors are mounted in such a way that the vehicle center of buoyancy (CB) and center of gravity (CG) are coincident (see Figure 2.1). Let the diagonal matrix with diagonal elements (J_1^i, J_2^i, J_3^i) denote the inertia matrix of the rotor which spins about the ith principal axis ($i = 1, 2,$ or 3). Assume that this rotor spins at a rate $\dot{\alpha}_i$ relative to the vehicle. Defining

$$\lambda_j = I_j + J_j^1 + J_j^2 + J_j^3, \qquad j = 1, 2, 3$$

gives the elements of the locked inertia matrix: $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$. Let $\boldsymbol{J_r} = \mathrm{diag}(J_1^1, J_2^2, J_3^3)$ be the matrix of rotor spin axis moments of inertia and define $\bar{\boldsymbol{I}} = \mathrm{diag}(\bar{I}_1, \bar{I}_2, \bar{I}_3) = \boldsymbol{\Lambda} - \boldsymbol{J_r}$.

The total kinetic energy of the vehicle/fluid system is

$$T = \frac{1}{2}\left(\boldsymbol{v} \cdot \boldsymbol{M}\boldsymbol{v} + \boldsymbol{\Omega} \cdot \bar{\boldsymbol{I}}\boldsymbol{\Omega} + (\boldsymbol{\Omega} + \boldsymbol{\Omega_r}) \cdot \boldsymbol{J_r}(\boldsymbol{\Omega} + \boldsymbol{\Omega_r})\right)$$

where $\boldsymbol{\Omega_r} = [\dot{\alpha}_1,\ \dot{\alpha}_2,\ \dot{\alpha}_3]^T$. The body coordinate momenta are

$$\begin{aligned} \boldsymbol{\Pi} &= \boldsymbol{\Lambda}\boldsymbol{\Omega} + \boldsymbol{J_r}\boldsymbol{\Omega_r}, \\ \boldsymbol{P} &= \boldsymbol{M}\boldsymbol{v}, \\ \boldsymbol{l} &= \boldsymbol{J_r}(\boldsymbol{\Omega} + \boldsymbol{\Omega_r}). \end{aligned}$$

$\boldsymbol{\Pi}$ and $\boldsymbol{P}$ are the total (body + rotors + fluid) angular and linear momentum, respectively. The ith element of $\boldsymbol{l}$ is the total momentum of the ith rotor about its spin axis. The control will be $\boldsymbol{u} = [u_1, u_2, u_3]^T$ where u_i is the torque applied to the ith internal rotor about its spin axis.

Rotational and translational viscous drag are included in the vehicle model. We assume that the damping torque takes the form $\boldsymbol{f_\Omega}(\boldsymbol{\Omega})$ where $\boldsymbol{f_\Omega}(\cdot)$ is a continuous function and $\boldsymbol{f_\Omega}(\boldsymbol{\Omega}) = \boldsymbol{0}$ if and only if $\boldsymbol{\Omega} = \boldsymbol{0}$. An additional assumption on angular damping is that

$$\|\boldsymbol{f_\Omega}(\boldsymbol{\Omega})\| \leq \bar{f}_\Omega(\|\boldsymbol{\Omega}\|) \tag{2.1}$$

where $\bar{f}_\Omega(\cdot)$ is some real analytic function for which $\bar{f}_\Omega(\|\boldsymbol{\Omega}\|) \geq 0$ with equality only when $\|\boldsymbol{\Omega}\| = 0$. The damping force is given by $\boldsymbol{f_v}(\boldsymbol{v})$ where $\boldsymbol{f_v}(\cdot)$ is continuous and $\boldsymbol{f_v}(\boldsymbol{v}) = \boldsymbol{0}$ if and only if $\boldsymbol{v} = \boldsymbol{0}$.

For example, a simple drag model given in [4] is

$$\begin{aligned} \boldsymbol{e_i} \cdot \boldsymbol{f_\Omega}(\boldsymbol{\Omega}) &= -(a_i + \bar{a}_i|\Omega_i|)\Omega_i \\ \boldsymbol{e_i} \cdot \boldsymbol{f_v}(\boldsymbol{v}) &= -(b_i + \bar{b}_i|v_i|)v_i \end{aligned}$$

where $\boldsymbol{e_1}, \boldsymbol{e_2}$, and $\boldsymbol{e_3}$ are standard basis vectors for $\mathcal{R}^3$ and all coefficients are positive constants. Then,

$$\|\boldsymbol{f_\Omega}(\boldsymbol{\Omega})\| \leq \max_i a_i\|\Omega\| + \max_i \bar{a}_i\|\Omega\|^2$$

and assumption (2.1) is satisfied.

Drag always opposes velocity. We further assume that drag grows at least linearly in velocity,

$$\begin{aligned} \Omega_i\ \boldsymbol{e_i} \cdot \boldsymbol{f_\Omega}(\boldsymbol{\Omega}) \leq -\underline{f}_{\Omega_i}\Omega_i^2 \ < \ 0 \quad (\Omega_i \neq 0) \\ v_i\ \boldsymbol{e_i} \cdot \boldsymbol{f_v}(\boldsymbol{v}) \leq -\underline{f}_{v_i}v_i^2 \ < \ 0 \quad (v_i \neq 0) \end{aligned} \tag{2.2}$$

where $\underline{f}_{\Omega_i}$ and $\underline{f}_{v_i}$ are constant, positive scalars. For the example drag model, one could choose any $\underline{f}_{\Omega_i}$ satisfying $0 < \underline{f}_{\Omega_i} \leq a_i$ and similarly for $\underline{f}_{v_i}$.

We make the additional assumption that, for any real scalar c,

$$\boldsymbol{e_i} \cdot \boldsymbol{f_v}(c\,\boldsymbol{e_1}) = 0, \qquad i = 2, 3. \tag{2.3}$$

Equation (2.3) implies that the vehicle experiences no lift or side force when moving purely along its long axis. This is a reasonable assumption for a vehicle which is symmetric about its 1-2 and 1-3 planes. The assumption does not prohibit a symmetric wing or empennage.

We include in the model a constant, body-fixed propeller force $\boldsymbol{F}$ which is sufficient to maintain the desired equilibrium, steady translation along the long axis with $\boldsymbol{v} = \boldsymbol{v_e} = \tilde{v}_1\boldsymbol{e_1}$. This constant thrust is

$$\boldsymbol{F} = -\boldsymbol{f_v}(\boldsymbol{v_e}).$$

Assumption (2.3) implies that thrust is aligned with the vehicle long axis.

We restrict our choice of equilibrium speeds to $\tilde{v}_1$ such that, when $\boldsymbol{\Omega} = \boldsymbol{0}$ and $v_2 = v_3 = 0$,

$$(v_1 - \tilde{v}_1)\boldsymbol{e_1} \cdot (\boldsymbol{f_v}(v_1\boldsymbol{e_1}) - \boldsymbol{f_v}(\tilde{v}_1\boldsymbol{e_1})) \le 0 \tag{2.4}$$

with equality if and only if $v_1 = \tilde{v}_1$. The assumption (2.4) requires that, when the vehicle translates along its long axis, the magnitude of drag is larger (smaller) than the magnitude of thrust when the vehicle moves faster (slower) than $\tilde{v}_1$. For the example drag model, there is no restriction on the choice of $\tilde{v}_1$. One might expect a small range of inadmissible equilibrium speeds in the neighborhood of the critical speed for boundary layer transition [5].

The equations of motion are

$$\begin{aligned}
\dot{\boldsymbol{\Pi}} &= \boldsymbol{\Pi} \times \boldsymbol{\Omega} + \boldsymbol{P} \times \boldsymbol{v} + \boldsymbol{f_\Omega}(\boldsymbol{\Omega}) \\
\dot{\boldsymbol{P}} &= \boldsymbol{P} \times \boldsymbol{\Omega} + \boldsymbol{f_v}(\boldsymbol{v}) - \boldsymbol{f_v}(\boldsymbol{v_e}) \\
\dot{\boldsymbol{l}} &= \boldsymbol{u}
\end{aligned} \tag{2.5}$$

where $\boldsymbol{\Omega} = \bar{\boldsymbol{I}}^{-1}(\boldsymbol{\Pi} - \boldsymbol{l})$ and $\boldsymbol{v} = \boldsymbol{M}^{-1}\boldsymbol{P}$.

3 Hamiltonian Stabilization

In this section we review our earlier stabilization work described in [11] where we ignore fluid drag. In this case, we consider our vehicle model (2.5) with $\boldsymbol{f_\Omega}(\cdot)$ and $\boldsymbol{f_v}(\cdot)$ set identically to zero.

The control $\boldsymbol{u}$ in equations (2.5) is prescribed as the sum of a term which treats the conservative part of the system and a dissipative term $\boldsymbol{u_d}$. The former term is chosen so that, with the dissipative term set to zero, the closed-loop system is Hamiltonian. Control parameters for this first term are then chosen to shape the kinetic energy and energy methods are used to prove closed-loop stability. The choice of the energy-shaping feedback control is inspired by the study of satellite control using internal rotors in [1]. It may also be derived using the method of controlled Lagrangians [2, 3].

Define the feedback control law

$$\boldsymbol{u} = \boldsymbol{K}\dot{\boldsymbol{\Pi}} + (\boldsymbol{\mathcal{I}} - \boldsymbol{K})\boldsymbol{u_d} \tag{3.1}$$

where $\boldsymbol{K} = \mathrm{diag}(k_1, k_2, k_3)$ is a matrix of control gains, $\boldsymbol{\mathcal{I}}$ is the 3×3 identity matrix and $\boldsymbol{u_d}$ is the dissipative control term to be determined. Make the change of variables $(\boldsymbol{\Pi}, \boldsymbol{P}, \boldsymbol{l}) \to (\boldsymbol{\Pi}, \boldsymbol{P}, \boldsymbol{\zeta})$ where

$$\boldsymbol{\zeta} = (\boldsymbol{\mathcal{I}} - \boldsymbol{K})^{-1}(\boldsymbol{l} - \boldsymbol{K}\boldsymbol{\Pi}). \tag{3.2}$$

The equations of motion (2.5) in the new variables are

$$\begin{aligned}
\dot{\boldsymbol{\Pi}} &= \boldsymbol{\Pi} \times \boldsymbol{\Omega} + \boldsymbol{P} \times \boldsymbol{v} + \boldsymbol{f_\Omega}(\boldsymbol{\Omega}) \\
\dot{\boldsymbol{P}} &= \boldsymbol{P} \times \boldsymbol{\Omega} + \boldsymbol{f_v}(\boldsymbol{v}) - \boldsymbol{f_v}(\boldsymbol{v_e}) \\
\dot{\boldsymbol{\zeta}} &= \boldsymbol{u_d}.
\end{aligned} \tag{3.3}$$

We now have

$$\boldsymbol{\Omega} = \boldsymbol{Z}^{-1}(\boldsymbol{\Pi} - \boldsymbol{\zeta}), \quad \boldsymbol{v} = \boldsymbol{M}^{-1}\boldsymbol{P}, \tag{3.4}$$

where the diagonal matrix $\boldsymbol{Z} = (\boldsymbol{\mathcal{I}} - \boldsymbol{K})^{-1}\bar{\boldsymbol{I}}$ is referred to as the "controlled inertia matrix" since it depends on the control gain matrix $\boldsymbol{K}$ and plays the role of inertia in the closed-loop system.

With fluid drag still neglected and $\boldsymbol{u_d} = \boldsymbol{0}$, the closed-loop equations (3.3) describe Hamiltonian dynamics with conserved Hamiltonian

$$H_C(\boldsymbol{\Pi}, \boldsymbol{P}, \boldsymbol{\zeta}) = \frac{1}{2}\boldsymbol{P}^T\boldsymbol{M}^{-1}\boldsymbol{P} + \frac{1}{2}(\boldsymbol{\Pi} - \boldsymbol{\zeta})^T\boldsymbol{Z}^{-1}(\boldsymbol{\Pi} - \boldsymbol{\zeta}). \tag{3.5}$$

Furthermore, inertial linear and angular momentum are conserved. In body coordinates, these conservation laws are reflected by the conserved quantities

$$C_1 = \frac{1}{2}\boldsymbol{P} \cdot \boldsymbol{P}, \quad \text{and} \quad C_2 = \boldsymbol{\Pi} \cdot \boldsymbol{P}.$$

Let $\boldsymbol{\Omega} = \boldsymbol{\Omega_e}$ and $\boldsymbol{v} = \boldsymbol{v_e}$ correspond to steady translation along the vehicle's long axis:

$$\boldsymbol{\Omega_e} = \boldsymbol{0}, \quad \boldsymbol{v_e} = \tilde{v}_1\boldsymbol{e_1}. \tag{3.6}$$

For a vehicle without internal rotors, this motion is an unstable relative equilibrium [8, 9].

It was shown in [11] that, when thrust and drag are absent, equilibria of the form (3.6) may be asymptotically stabilized using internal rotors. The proof of stability involves using the energy-Casimir method to construct a conserved Lyapunov function H_Φ for which the equilibrium is a relative *maximum*. The condition on the control gain matrix $\boldsymbol{K}$ is that $\boldsymbol{\mathcal{I}} - \boldsymbol{K} < 0$ so that $\boldsymbol{Z} < 0$. The equilibrium may be made asymptotically stable by choosing feedback dissipation $\boldsymbol{u_d}$ such that $\frac{d}{dt}H_\Phi \ge 0$.

4 Feedback Dissipation

In this section, we preserve the conservative part of the control law but amend the dissipative control term $\boldsymbol{u_d}$

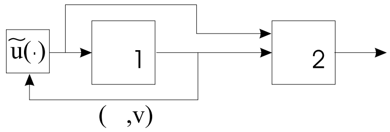

Figure 4.1: Interconnection of Σ_1 and Σ_2.

in order to provide global asymptotic stabilization of the desired equilibrium motion in the presence of fluid drag. Since the desired equilibrium is a maximum of the Lyapunov function for the conservative part of the system and drag acts to decrease the system energy, the fluid drag model as described in Section 2 is destabilizing. In addition, drag makes $\frac{d}{dt}H_\Phi$ indefinite.

To find conditions for stability of long axis translation in the presence of thrust and drag, we break the system into two subsystems. The first subsystem is described solely in terms of the vehicle velocities while the second subsystem involves the rotor angular velocities, as well. Using a modified version of H_Φ and an argument based on LaSalle's invariance principle, we prove global asymptotic stability of the first subsystem to the desired state (3.6). The state of the second subsystem is shown to be bounded implying that the rotor angular velocities are bounded.

Using (3.4) to transform variables from $(\boldsymbol{\Pi}, \boldsymbol{P}, \boldsymbol{\zeta})$ to $(\boldsymbol{\Omega}, \boldsymbol{v}, \boldsymbol{\zeta})$, the equations of motion (3.3) become

$$\begin{aligned}
\dot{\boldsymbol{\Omega}} &= \boldsymbol{Z}^{-1}\left[(\boldsymbol{Z\Omega}+\boldsymbol{\zeta})\times\boldsymbol{\Omega}+\boldsymbol{Mv}\times\boldsymbol{v}+\boldsymbol{f_\Omega}(\boldsymbol{\Omega})-\boldsymbol{u_d}\right]\\
\dot{\boldsymbol{v}} &= \boldsymbol{M}^{-1}\left[\boldsymbol{Mv}\times\boldsymbol{\Omega}+\boldsymbol{f_v}(\boldsymbol{v})-\boldsymbol{f_v}(\boldsymbol{v_e})\right]\\
\dot{\boldsymbol{\zeta}} &= \boldsymbol{u_d}.
\end{aligned}$$

Defining

$$\boldsymbol{u_d} = \boldsymbol{\zeta}\times\boldsymbol{\Omega}+\tilde{\boldsymbol{u}} \tag{4.1}$$

where $\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{u}}(\boldsymbol{\Omega}, \boldsymbol{v})$ makes the $\dot{\boldsymbol{\Omega}}$ and $\dot{\boldsymbol{v}}$ equations independent of $\boldsymbol{\zeta}$. The system may then be broken into two subsystems. The first subsystem, Σ_1, describes the $(\boldsymbol{\Omega}, \boldsymbol{v})$ dynamics:

$$\begin{aligned}
\dot{\boldsymbol{\Omega}} &= \boldsymbol{Z}^{-1}\left[\boldsymbol{Z\Omega}\times\boldsymbol{\Omega}+\boldsymbol{Mv}\times\boldsymbol{v}+\boldsymbol{f_\Omega}(\boldsymbol{\Omega})-\tilde{\boldsymbol{u}}\right]\\
\dot{\boldsymbol{v}} &= \boldsymbol{M}^{-1}\left[\boldsymbol{Mv}\times\boldsymbol{\Omega}+\boldsymbol{f_v}(\boldsymbol{v})-\boldsymbol{f_v}(\boldsymbol{v_e})\right].
\end{aligned} \tag{4.2}$$

The second subsystem, Σ_2, describes the $\boldsymbol{\zeta}$ dynamics driven by $\boldsymbol{\Omega}$ and $\boldsymbol{v}$:

$$\dot{\boldsymbol{\zeta}} = -\boldsymbol{\Omega}\times\boldsymbol{\zeta}+\tilde{\boldsymbol{u}}. \tag{4.3}$$

Figure 4.1 depicts the interconnection of Σ_1 and Σ_2.

We first prescribe $\tilde{\boldsymbol{u}}$ and prove stability of Σ_1 to the desired equilibrium (3.6).

Define the negative semidefinite function

$$V = \frac{1}{2}\boldsymbol{v}^T\boldsymbol{M}(\boldsymbol{M}^{-1}-\frac{1}{m_1}\boldsymbol{\mathcal{I}})\boldsymbol{Mv}+\frac{1}{2}\boldsymbol{\Omega}^T\boldsymbol{Z\Omega} \tag{4.4}$$

by truncating H_Φ from [11]. By (4.2),

$$\begin{aligned}
\dot{V}(\boldsymbol{\Omega}, \boldsymbol{v}) \;=\; &\sum_{i=2}^{3}\frac{(m_1-m_i)}{m_1}v_i\,\boldsymbol{e_i}\cdot(\boldsymbol{f_v}(\boldsymbol{v})-\boldsymbol{f_v}(\boldsymbol{v_e}))+\\
&\boldsymbol{\Omega}\cdot(\boldsymbol{f_\Omega}(\boldsymbol{\Omega})-\tilde{\boldsymbol{u}}).
\end{aligned} \tag{4.5}$$

Define the dissipative feedback with control gain δ as

$$\tilde{\boldsymbol{u}} = \delta\,\boldsymbol{f_\Omega}(\boldsymbol{\Omega}), \qquad \delta > 1. \tag{4.6}$$

Then, under the assumptions (2.2) and (2.3) on the form of drag, $\dot{V}$ given by (4.5) is positive semidefinite:

$$\dot{V} \geq \sum_{i=2}^{3}\frac{(m_i-m_1)}{m_1}\underline{f}_{v_i}v_i^2+(\delta-1)\sum_{j=1}^{3}\underline{f}_{\Omega_j}\Omega_j^2 \tag{4.7}$$

and $\dot{V}=0$ when $\boldsymbol{\Omega}$, v_2, and v_3 are all zero.

To conclude stability of the desired equilibrium using LaSalle's invariance principle, we must first find a trapping region containing the equilibrium. The set

$$\mathcal{T}_1 = \{(\boldsymbol{\Omega}, \boldsymbol{v}) \mid V \geq -c_1\}, \qquad c_1 > 0$$

is positively invariant, but it is not compact because v_1 is not represented in V.

To define a valid trapping region, consider the effect of thrust and drag on the translational momentum. The rate of change of $C_1 = \frac{1}{2}\|\boldsymbol{P}\|^2$ is

$$\frac{d}{dt}\left(\frac{1}{2}\|\boldsymbol{P}\|^2\right) = \boldsymbol{P}\cdot\dot{\boldsymbol{P}} = \boldsymbol{P}\cdot(\boldsymbol{f_v}(\boldsymbol{v})-\boldsymbol{f_v}(\boldsymbol{v_e})).$$

Using (2.2), it follows that

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{2}\|\boldsymbol{P}\|^2\right) \;&\leq\; -\sum_{i=1}^{3}m_i\underline{f}_{v_i}v_i^2-\boldsymbol{P}\cdot\boldsymbol{f_v}(\boldsymbol{v_e})\\
&\leq\; -\min_i\left(\frac{\underline{f}_{v_i}}{m_i}\right)\|\boldsymbol{P}\|^2+\|\boldsymbol{f_v}(\boldsymbol{v_e})\|\|\boldsymbol{P}\|.
\end{aligned}$$

Thus if $\|\boldsymbol{P}\|$ is large, C_1 is decreasing. In other words, there is some maximum speed which the vehicle can sustain. Choose

$$c_2 \geq \frac{1}{2}\left(\frac{\|\boldsymbol{f_v}(\boldsymbol{v_e})\|}{\min_i\left(\frac{\underline{f}_{v_i}}{m_i}\right)}\right)^2.$$

Then another positively invariant set is

$$\mathcal{T}_2 = \left\{\boldsymbol{P}\;\middle|\;\frac{1}{2}\|\boldsymbol{P}\|^2 \leq c_2\right\}.$$

$\mathcal{T} = \mathcal{T}_1\bigcap\mathcal{T}_2$ is compact, positively invariant, and contains the equilibrium (3.6). Define

$$E = \{(\boldsymbol{\Omega}, \boldsymbol{v}) \in \mathcal{T} \mid \boldsymbol{\Omega}=0, v_2=v_3=0\} \subset \mathcal{R}.$$

If $\mathcal{M}$ is the largest positively invariant set contained in E then solutions starting in $\mathcal{T}$ go to $\mathcal{M}$ by LaSalle's invariance principle (see [7], for example). The only nonzero state within E is v_1. By assumption (2.3), $\mathcal{M} = E$ is an invariant surface whose dynamics are described by

$$\dot{v}_1 = \frac{1}{m_1}\boldsymbol{e_1}\cdot(\boldsymbol{f_v}(v_1\boldsymbol{e_1}) - \boldsymbol{f_v}(\tilde{v}_1\boldsymbol{e_1}))\,.$$

Under assumption (2.4), $\frac{1}{2}(v_1-\tilde{v}_1)^2$ is a Lyapunov function on $\mathcal{M}$ and v_1 goes to $\tilde{v}_1$ asymptotically on this invariant surface.

All trajectories go to $\mathcal{M}$ and all trajectories starting on $\mathcal{M}$ go to the desired equilibrium. Since trajectories close to $\mathcal{M}$ are dominated by the dynamics on $\mathcal{M}$, we conclude that $\boldsymbol{\Omega} \to \mathbf{0}$ and $\boldsymbol{v} \to \boldsymbol{v_e}$ asymptotically. Furthermore, this is true *globally.*

We may estimate the rate of convergence of the Lyapunov function. First, we rewrite V as

$$V = \frac{1}{2}\left(\frac{1}{L_1}\boldsymbol{v}^T\right)\boldsymbol{M}L_1^2(\boldsymbol{\mathcal{I}} - \frac{1}{m_1}\boldsymbol{M})\left(\frac{1}{L_1}\boldsymbol{v}\right) + \frac{1}{2}\boldsymbol{\Omega}^T\boldsymbol{Z}\boldsymbol{\Omega}$$

where L_1 is a characteristic length. Let $\boldsymbol{\xi}^T = [\frac{1}{L_1}v_2, \frac{1}{L_1}v_3, \boldsymbol{\Omega}^T]$. Then,

$$0 \geq V \geq -b\|\boldsymbol{\xi}\|^2,$$

$$b = \max\{(\frac{1}{m_1} - \frac{1}{m_3})(m_3L_1)^2, -Z_1, -Z_2, -Z_3\} > 0.$$

By equation (4.7),

$$\dot{V} \geq a\|\boldsymbol{\xi}\|^2,$$

$$a = \min\left(L_1^2\min_{i=2,3}(\frac{1}{m_1} - \frac{1}{m_i})m_i\underline{f}_{v_i}, (\delta-1)\min_j \underline{f}_{\Omega_j}\right) > 0.$$

Therefore,

$$\dot{V} \geq \frac{a}{b}\left(b\|\boldsymbol{\xi}\|^2\right) \geq -\left(\frac{a}{b}\right)V$$

and V grows toward zero exponentially,

$$0 \geq V(t) \geq V(0)e^{-\frac{a}{b}t}. \tag{4.8}$$

Having bounded V we may bound $\boldsymbol{\Omega}$. Let $\underline{Z}$ denote the magnitude of the least negative eigenvalue of $\boldsymbol{Z}$, that is $\underline{Z} = \min_i|Z_i|$. Then,

$$\|\boldsymbol{\Omega}\|^2 \leq \frac{2}{\underline{Z}}|\frac{1}{2}\boldsymbol{\Omega}^T\boldsymbol{Z}\boldsymbol{\Omega}| \leq \frac{2}{\underline{Z}}|V| \leq \frac{2}{\underline{Z}}|V(0)|e^{-\frac{a}{b}t}.$$

Thus $\|\boldsymbol{\Omega}\|$ decays exponentially:

$$\|\boldsymbol{\Omega}\| \leq Xe^{-\frac{1}{2}(\frac{a}{b})t} \tag{4.9}$$

where $X = \sqrt{2|V(0)|/\underline{Z}}$.

By equation (4.3) with $\tilde{\boldsymbol{u}}$ given by equation (4.6), $\dot{\boldsymbol{\zeta}}$ goes to zero as $\boldsymbol{\Omega}$ goes to zero. Note that

$$\begin{aligned}\|\boldsymbol{\zeta}\|\frac{d}{dt}\|\boldsymbol{\zeta}\| &= \frac{d}{dt}\left(\frac{1}{2}\|\boldsymbol{\zeta}\|^2\right)\\ &= \boldsymbol{\zeta}\cdot(\boldsymbol{\zeta}\times\boldsymbol{\Omega} + \delta\boldsymbol{f_\Omega}(\boldsymbol{\Omega}))\\ &\leq \delta\|\boldsymbol{\zeta}\|\|\boldsymbol{f_\Omega}(\boldsymbol{\Omega})\|.\end{aligned} \tag{4.10}$$

By assumption (2.1), there is a real analytic function $\bar{f}_\Omega(\cdot)$ such that $\|\boldsymbol{f_\Omega}(\boldsymbol{\Omega})\| \leq \bar{f}_\Omega(\|\boldsymbol{\Omega}\|)$ and $\bar{f}_\Omega(0) = 0$. It follows that $\|\boldsymbol{\zeta}\|$ is bounded. We compute the bound in the case that

$$\|\boldsymbol{f_\Omega}(\boldsymbol{\Omega})\| \leq \tilde{f}_\Omega\|\boldsymbol{\Omega}\|$$

where $\tilde{f}_\Omega$ is a positive scalar. By (4.10),

$$\frac{d}{dt}\|\boldsymbol{\zeta}\| \leq \delta\tilde{f}_\Omega\|\boldsymbol{\Omega}\|$$

whenever $\|\boldsymbol{\zeta}\| \neq 0$. Using (4.9) and integrating the above inequality from 0 to $t \geq 0$, one obtains

$$\|\boldsymbol{\zeta}(t)\| \leq 2\delta\tilde{f}_\Omega\left(\frac{b}{a}\right)X + \|\boldsymbol{\zeta}(0)\|. \tag{4.11}$$

To minimize the bound on $\|\boldsymbol{\zeta}\|$ for a given drag model, we should choose control parameters $\boldsymbol{K}$ and δ such that $\delta\left(\frac{b}{a}\right)/\sqrt{\underline{Z}}$ is small. From the definitions of a and b, the ratio $\frac{b}{a}$ is smallest when

$$|Z_i| < \left(\frac{1}{m_1} - \frac{1}{m_3}\right)(m_3L_1)^2 \qquad i = 1,2,3 \tag{4.12}$$

and

$$\delta > 1 + \frac{L_1^2\min_{i=2,3}\left\{(\frac{1}{m_1} - \frac{1}{m_i})m_i\underline{f}_{v_i}\right\}}{\min_j\left\{\underline{f}_{\Omega_j}\right\}}. \tag{4.13}$$

Note that choosing the control such that $\frac{b}{a}$ is as small as possible also maximizes the rate at which $\|\boldsymbol{\Omega}\|$ converges to zero. Contrary to conditions (4.12) and (4.13), however, $\delta/\sqrt{\underline{Z}}$ is smallest when we choose $0 < \delta - 1 \ll 1$ and $|Z_i|$ large for $i = 1, 2,$ and 3. A reasonable compromise is to choose the control so that conditions (4.12) and (4.13) are just satisfied. This choice will ensure that $\|\boldsymbol{\Omega}\|$ converges as rapidly as possible (with the convergence rate dictated by the fluid drag) and that $\delta/\sqrt{\underline{Z}}$ is as small as possible, given $\frac{b}{a}$.

Converting back to $(\boldsymbol{\Pi}, \boldsymbol{P}, \boldsymbol{\zeta})$ variables, the equations of motion with the given dissipative feedback are

$$\begin{aligned}\dot{\boldsymbol{\Pi}} &= \boldsymbol{\Pi}\times\boldsymbol{\Omega} + \boldsymbol{P}\times\boldsymbol{v} + \boldsymbol{f_\Omega}(\boldsymbol{\Omega})\\ \dot{\boldsymbol{P}} &= \boldsymbol{P}\times\boldsymbol{\Omega} + \boldsymbol{f_v}(\boldsymbol{v}) - \boldsymbol{f_v}(\boldsymbol{v_e})\\ \dot{\boldsymbol{\zeta}} &= \boldsymbol{\zeta}\times\boldsymbol{\Omega} + \delta\boldsymbol{f_\Omega}(\boldsymbol{\Omega}).\end{aligned} \tag{4.14}$$

In these coordinates, an equilibrium $(\boldsymbol{\Omega_e}, \boldsymbol{v_e}, \boldsymbol{\zeta}_e) = (\mathbf{0}, \tilde{v}_1\boldsymbol{e_1}, \tilde{\boldsymbol{\Pi}})$ becomes $(\boldsymbol{\Pi_e}, \boldsymbol{P_e}, \boldsymbol{\zeta}_e) = (\tilde{\boldsymbol{\Pi}}, m_1\tilde{v}_1\boldsymbol{e_1}, \tilde{\boldsymbol{\Pi}})$

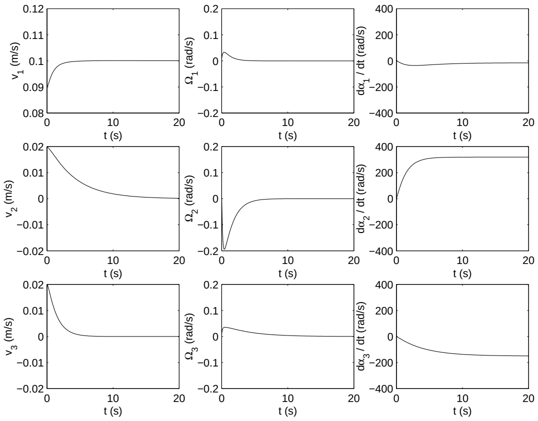

Figure 4.2: Closed-loop response to a perturbation.

Theorem 4.1 *Equations (4.14) with* $\delta > 1$ *describe a system whose state goes asymptotically to an equilibrium*

$$\boldsymbol{\Pi_e} = \tilde{\boldsymbol{\Pi}}, \quad \boldsymbol{P_e} = \boldsymbol{M}\boldsymbol{v_e}, \quad \boldsymbol{\zeta}_e = \tilde{\boldsymbol{\Pi}}$$

regardless of initial condition.

The final equilibrium value of $\boldsymbol{\Pi}$ and $\boldsymbol{\zeta}$ will vary with initial condition but the vehicle linear and angular velocity will always approach the desired values. When $\boldsymbol{\Omega}$ is zero, $\boldsymbol{\zeta}$ corresponds to the rotor angular momentum. While we do not expect to drive the internal rotors to zero angular velocity with this choice of control, we know the rotor angular velocities will be bounded. Ensuring that rotor angular velocities remain within practical limits is an issue that warrants further attention. Continuing work involves extending the model to the case of a vehicle with a low center of gravity which should lessen the demands on the internal rotors.

Numerical simulations were performed for a vehicle modeled as an ellipsoid with axis lengths $L_1 = 0.4572$ m, $L_2 = 0.3048$ m and $L_3 = 0.1524$ m. For a neutrally buoyant body of this shape, the mass plus added mass terms are $m_1 = 13.2$ kg, $m_2 = 15.2$ kg and $m_3 = 25.6$ kg. Each "rotor" is modeled as a pair of parallel, coupled thin disks each of mass $m_{disk} = 1$ kg and radius $r = 0.0508$ m. Each of these two disks has its spin axis aligned with a given principal axis and is located $d = 0.0572$ m along the principal axis from the vehicle CB, one in either direction.

Drag was modeled according to the example in Section 2 with $a_i = 1$ Ns, $b_i = 1$ Ns/m, and $\bar{a}_i = \bar{b}_i = 0$ for $i = 1, 2$, and 3. These values yield moments and forces of the appropriate order for a vehicle of the given size moving at 0.1 m/s.

It is desired to stabilize this vehicle with no body angular velocity, $\boldsymbol{\Omega_e} = \mathbf{0}$, and a forward speed $\tilde{v}_1 = 0.1$ m/s. The control gains are chosen to satisfy the requirements developed in the preceding analysis. Specifically, we choose $\boldsymbol{K} = \mathrm{diag}(3.9, 8.3, 10)$ so that $\boldsymbol{Z} = \mathrm{diag}(-0.0171, -0.0127, -0.0112)$ and we take $\delta = 1.1$.

Figure 4.2 shows the body angular and linear velocity response and the rotor angular rate response to an initial perturbation from the desired equilibrium. Initial conditions are $\boldsymbol{\Omega}(0) = [0.01, 0.01, 0.01]^T$ rad/s, $\boldsymbol{v}(0) = [0.09, 0.02, 0.02]^T$ m/s, and $\boldsymbol{\Omega_r}(0) = [1, 1, 1]^T$ rad/s. The velocities $\boldsymbol{v}$ and $\boldsymbol{\Omega}$ approach the desired values while each rotor's angular velocity approaches a nonzero constant, as expected.

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