Automatica, Vo. 38, No. 12, pp. 2053-2062.

Stabilizing Underwater Vehicle Motion Using Internal Rotors ⋆

C. A. Woolsey [a,1] and N. E. Leonard [b,2]

[a] *Aerospace & Ocean Engineering, Virginia Tech, Blacksburg, VA 24061 USA*

[b] *Mechanical & Aerospace Engineering, Princeton University, Princeton, NJ 08544 USA*

Abstract

As a case study of a particular control methodology and as a practical contribution in the area of underwater vehicle control, we consider the problem of stabilizing an underwater vehicle using internal rotors as actuators. The control design method comprises three steps. The first step involves shaping the kinetic energy of the conservative dynamics. For the underwater vehicle, the control term from this step may be interpreted as modifying the system inertia. In the second step, we design feedback dissipation using a Lyapunov function constructed in the first step. In the third step, we include a general model for the viscous force and moment on the vehicle and we show that these effects enhance stability. We first apply this method to a vehicle whose centers of buoyancy and gravity coincide and then to a vehicle with noncoincident centers of buoyancy and gravity.

Key words: underwater vehicle, reaction wheel, Hamiltonian systems, stabilization, energy shaping

1 Introduction

Underwater vehicles (UV's) are enjoying a great deal of attention from ocean scientists, who envision cost-effective mobile sensor arrays, and from control theorists, who see a rich test bed for control design techniques developed for mechanical systems. This paper serves both communities by expanding the performance capabilities for a class of UV's and by demonstrating the tools available from geometric mechanics for the design of stabilizing control laws.

Traditional UV actuators, such as propellers and fins, provide adequate control authority over a range of operating conditions. However, increasingly ambitious requirements are pushing the conventional performance envelope. For example, the practical implementation of a proposed long-term, remote ocean sensing network (Curtin, Bellingham, Catipovic, & Webb, 1993) will require efficient, durable, and maneuverable vehicles which can resist or tolerate impairments that are often caused by the harsh ocean environment.

Typically, UV's are designed with a slender hull, a rear thruster, and actuated tail fins. A drawback of this arrangement is that fin actuators lose control authority at low velocity. While additional thrusters might be added to provide low-velocity control, these would increase the drag on the vehicle, reducing its efficiency and, therefore, its endurance. Furthermore, actuated fins and thrusters are subject to corrosion and biological fouling.

The inherent limitations of conventional actuators lead one to consider internal actuators as an alternative or complementary means of control. Internal actuators do not rely on relative fluid motion to exert control, so they add no drag and they are useful even at zero velocity, significantly extending a vehicle's operating range. Internal actuators also preserve the integrity of the vehicle housing because no wiring, cables, or drive shafts penetrate the hull. Internal moving masses are already used to provide attitude control for a number of UV's, notably for

---

⋆ This paper was not presented at an IFAC meeting.

Email addresses: `cwoolsey@vt.edu`, `naomi@princeton.edu` (N. E. Leonard).

[1] Corresponding author. Tel: +1-540-231-8117; Fax: +1-540-231-9632.

[2] Research partially supported by the National Science Foundation under grant BES-9502477 and by the Office of Naval Research under grants N00014-96-1-0052 and N00014-98-1-0649.

underwater *gliders*; see (Leonard & Graver, 2001) for examples and references. Alternatively, we have proposed using internal rotors to stabilize steady, long-axis translation of a slender UV (Leonard & Woolsey, 1998).

Building on earlier work (Woolsey & Leonard, 1999a,b), we present a three-step approach to design a control law that stabilizes a UV using internal rotors:

1. Apply feedback which preserves the Hamiltonian structure of an idealized model and which shapes kinetic energy to stabilize a desired steady motion.
2. Use a Lyapunov function developed in Step 1 to design feedback dissipation to asymptotically stabilize the motion.
3. Examine the effect of physical damping on stability. If necessary, modify the feedback dissipation to ensure asymptotic stability.

Hamiltonian control systems were introduced by Brockett (1976). Early work on stabilizing equilibria for Hamiltonian control systems focused on shaping the closed-loop potential energy (van der Schaft, 1986). More recent work on potential energy shaping has provided extensions to underactuated systems as in, for example, (Jalnapurkar & Marsden, 2000; Bullo, 2000). However, the class of underactuated systems which can be stabilized through potential shaping alone is limited. In fact, one may also use feedback to shape the kinetic energy. The idea of kinetic energy shaping was suggested by Bloch, Krishnaprasad, Marsden, & Sánchez (1992), who extended the work of Krishnaprasad (1985) on spacecraft stabilization using internal rotors. More recently, the method of controlled Lagrangians has made this approach algorithmic for a class of mechanical systems (Bloch, Leonard, & Marsden, 2000; Bloch, Chang, Leonard, & Marsden, 2001a). One can view the energy-shaping step in this paper as an application of this method specialized to the subclass of Euler-Poincaré (reduced Lagrangian) systems (Bloch, Leonard, & Marsden, 2001b). Rather than treat the UV in a Lagrangian setting, however, we study the Hamiltonian form of the dynamics. An alternate approach to kinetic shaping in the Hamiltonian setting is interconnection and damping assignment, passivity-based control (IDA-PBC) (Ortega, Spong, Gómez-Estern, & Blankenstein, 2002; **?**). In their most general form, the method of controlled Lagrangians and IDA-PBC are equivalent (Chang, Bloch, Leonard, Marsden & Woolsey, 2002). We note that Astolfi, Chhabra, & Ortega (2001) have applied the IDA-PBC approach to a conventionally actuated UV.

In Section 2, we describe the dynamic equations which model the motion of a UV. In Section 3, we consider the problem of stabilizing steady long-axis translation of a vehicle whose center of gravity (CG) coincides with its center of buoyancy (CB). Section 4 treats a more general case of a vehicle whose CG is below its CB. We conclude in Section 5.

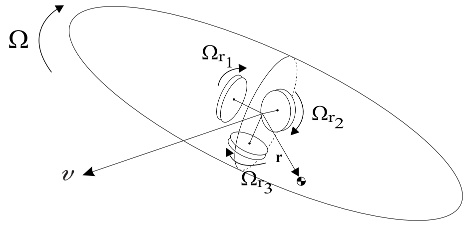

Fig. 1. A rigid body with rotors.

2 Underwater Vehicle Equations of Motion

In this section, we present a dynamic model for a UV with internal rotors. We begin with an idealized model and later introduce real effects such as viscous drag.

2.1 Rigid Body with Rotors in an Ideal Fluid

We model the UV as a rigid, non-axisymmetric ellipsoidal hull which houses three internal rotors. We fix a coordinate frame, described by the orthonormal vectors $\boldsymbol{e_1}$, $\boldsymbol{e_2}$, and $\boldsymbol{e_3}$, in the ellipsoid principal axes. The origin of this frame is at the vehicle's CB. Let L_i denote the length of the ith ellipsoid principal axis; without loss of generality, we may assume that $L_1 > L_2 > L_3$.

We assume that the hull's mass is distributed such that the inertia of the hull is diagonal, with respect to the coordinate axes. The internal rotors are mounted orthogonally within the vehicle so that each rotor's spin axis is aligned with a body coordinate axis. Each rotor spins about an axis of symmetry under the influence of a control torque. The CG of the internal rotor assembly is assumed to coincide with the body coordinate origin. The vector $\boldsymbol{r}$ denotes the CG of the complete vehicle with respect to the body coordinate origin. (See Figure 1.)

To obtain a preliminary dynamic model, we assume that the vehicle is immersed in an irrotational, incompressible, inviscid fluid which is at rest at the infinitely distant boundary. We also assume that the vehicle mass m is equal to the mass of fluid displaced by the vehicle; that is, we assume the vehicle is neutrally buoyant.

Let $\boldsymbol{\Omega}$ and $\boldsymbol{v}$ represent the angular and translational velocity of the vehicle with respect to inertial space, written in body coordinates. Let $\boldsymbol{\Omega_r} = (\Omega_{r_1}, \Omega_{r_2}, \Omega_{r_3})^T$ be the vector of rotor relative angular velocities, as in Figure 1. Define $\boldsymbol{\eta} = [\boldsymbol{\Omega}^T,\ \boldsymbol{v}^T,\ \boldsymbol{\Omega_r}^T]^T$. One may obtain the dynamic equations by treating the combined body/fluid/rotor system as a single dynamical system with kinetic energy

$$T = \frac{1}{2}\boldsymbol{\eta}\cdot\mathbb{M}\boldsymbol{\eta} \qquad \text{where} \qquad \mathbb{M} = \begin{pmatrix} \boldsymbol{\Lambda} & m\hat{\boldsymbol{r}} & \boldsymbol{J_r} \\ -m\hat{\boldsymbol{r}} & \boldsymbol{M} & \boldsymbol{0} \\ \boldsymbol{J_r} & \boldsymbol{0} & \boldsymbol{J_r} \end{pmatrix}.$$

Here, $\mathbb{M}$ is the generalized inertia for the body/fluid/rotor system. The sub-matrix $\mathbf{\Lambda} = \mathrm{diag}(\Lambda_1, \Lambda_2, \Lambda_3)$ is the inertia of the vehicle/fluid system with the rotors locked in place; $\mathbf{\Lambda}$ includes the *added inertia* from a potential flow model of the fluid motion (Lamb, 1932). Similarly, $\boldsymbol{M} = \mathrm{diag}(m_1, m_2, m_3)$ is the sum of $m\mathbb{I}$ and the *added mass* matrix (where $\mathbb{I}$ denotes the 3×3 identity matrix). $\boldsymbol{J_r} = \mathrm{diag}(J_1, J_2, J_3)$ is the matrix of rotor spin-axis moments of inertia; that is, J_i is the moment of inertia of the ith rotor about its spin axis. We let $\hat{\cdot}$ denote the skew-symmetric matrix satisfying $\hat{\boldsymbol{x}}\boldsymbol{y} = \boldsymbol{x} \times \boldsymbol{y}$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3$.

Because we have assumed that $L_1 > L_2 > L_3$, it follows that $m_1 < m_2 < m_3$; the added mass is least along the longest ellipsoid axis. The ordering of m_1, m_2, and m_3 plays a critical role in determining stability of steady motions. The ordering of the elements of the added inertia matrix depends on the relative lengths of the ellipsoid axes, as well as their ordering (Leonard, 1997a; Holmes, Jenkins, & Leonard, 1998).

Let $\boldsymbol{\nu} = \mathbb{M}\boldsymbol{\eta} = [\mathbf{\Pi}^T,\ \boldsymbol{P}^T,\ \boldsymbol{l}^T]^T$ represent the vector of conjugate momenta to $\mathbf{\Omega}$, $\boldsymbol{v}$, and $\mathbf{\Omega_r}$. In reality, $\mathbf{\Pi}$ and $\boldsymbol{P}$ are the angular and translational impulse necessary to generate the system motion from rest. Following Lamb (1932), we refer to $\mathbf{\Pi}$ and $\boldsymbol{P}$ as body angular momentum and body translational momentum, respectively. The vector $\boldsymbol{l}$ represents the total angular momentum of the internal rotors in body coordinates.

Suppose that $\boldsymbol{r} = \mathbf{0}$ so that the vehicle CG is located at the CB. Then the dynamic equations are

$$\begin{aligned}\dot{\mathbf{\Pi}} &= \mathbf{\Pi} \times \mathbf{\Omega} + \boldsymbol{P} \times \boldsymbol{v} + \boldsymbol{\mathcal{T}} \\ \dot{\boldsymbol{P}} &= \boldsymbol{P} \times \mathbf{\Omega} + \boldsymbol{\mathcal{F}} \\ \dot{\boldsymbol{l}} &= \boldsymbol{u},\end{aligned} \tag{1}$$

where $\boldsymbol{\mathcal{T}}$ and $\boldsymbol{\mathcal{F}}$ represent external torques and forces (e.g., viscous effects) and $\boldsymbol{u}$ represents internal control torques applied to the three internal rotors. With $\boldsymbol{\mathcal{T}}$, $\boldsymbol{\mathcal{F}}$, and $\boldsymbol{u}$ equal to zero, equations (1) define a Lie-Poisson (noncanonical Hamiltonian) system (Marsden & Ratiu, 1994). In this case, the dynamics can be expressed in terms of a *Poisson bracket* $\{\cdot,\cdot\}$ defined such that $\{G, K\}(\boldsymbol{\nu}) = \nabla G \cdot \mathbb{J}(\boldsymbol{\nu})\nabla K$, where $G(\boldsymbol{\nu})$ and $K(\boldsymbol{\nu})$ are differentiable functions and where

$$\mathbb{J} = \begin{pmatrix} \hat{\mathbf{\Pi}} & \hat{\boldsymbol{P}} & \mathbf{0} \\ \hat{\boldsymbol{P}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \tag{2}$$

is the skew-symmetric *Poisson tensor*. The Poisson tensor generalizes the symplectic matrix from classical mechanics. Let ν_i represent the ith component of $\boldsymbol{\nu}$ for $i \in \{1, ..., 9\}$. With $\boldsymbol{\mathcal{T}}$, $\boldsymbol{\mathcal{F}}$, and $\boldsymbol{u}$ zero, equations (1) are given by $\dot{\nu}_i = \{\nu_i, H\}$ for $i = 1, \ldots, 9$ or, in vector form,

$$\dot{\boldsymbol{\nu}} = \mathbb{J}\nabla H \qquad \text{where} \qquad H(\boldsymbol{\nu}) = \frac{1}{2}\boldsymbol{\nu} \cdot \mathbb{M}^{-1}\boldsymbol{\nu}.$$

There are five distinguished conserved quantities, called Casimir functions, in addition to the Hamiltonian: $C_1(\boldsymbol{\nu}) = \frac{1}{2}\boldsymbol{P} \cdot \boldsymbol{P}$, $C_2(\boldsymbol{\nu}) = \mathbf{\Pi} \cdot \boldsymbol{P}$, and each component of the rotor momentum $\boldsymbol{l}$. A Casimir function C *Poisson commutes* with any other smooth function K; that is, $\{C, K\} = 0$. Physically, the Casimirs C_1 and C_2 correspond, in body coordinates, to conservation of inertial linear and angular momentum, respectively; see (Leonard, 1997a). It is simple to verify that C_1 and C_2 are conserved for *any* choice of $\boldsymbol{u}$. This observation reflects the fact that internal actuators cannot affect inertial momentum.

In practical settings, a UV is trimmed so that the CG is below the CB for stability. Therefore, we also consider the more general case where $\boldsymbol{r} \neq \mathbf{0}$, i.e., where the vehicle's CB and CG do not coincide. Let $\mathbf{\Gamma}$ be the unit vector representing the direction of gravity in body coordinates. The equations of motion for a neutrally buoyant vehicle with $\boldsymbol{r} \neq \mathbf{0}$ are

$$\begin{aligned}\dot{\mathbf{\Pi}} &= \mathbf{\Pi} \times \mathbf{\Omega} + \boldsymbol{P} \times \boldsymbol{v} + \boldsymbol{r} \times mg\mathbf{\Gamma} + \boldsymbol{\mathcal{T}} \\ \dot{\boldsymbol{P}} &= \boldsymbol{P} \times \mathbf{\Omega} + \boldsymbol{\mathcal{F}} \\ \dot{\mathbf{\Gamma}} &= \mathbf{\Gamma} \times \mathbf{\Omega} \\ \dot{\boldsymbol{l}} &= \boldsymbol{u},\end{aligned} \tag{3}$$

where $\boldsymbol{\mathcal{T}}$ and $\boldsymbol{\mathcal{F}}$ are external torques and forces in addition to those due to buoyancy and gravity. Note that, although the gravitational force is balanced by the equal and opposite buoyant force, an external torque $\boldsymbol{r} \times mg\mathbf{\Gamma}$ acts about the body coordinate origin.

If $\boldsymbol{\mathcal{T}}$, $\boldsymbol{\mathcal{F}}$, and $\boldsymbol{u}$ are zero, then the dynamics (3) are Lie-Poisson. Let $\boldsymbol{\nu}_\Gamma = [\mathbf{\Pi}^T,\ \boldsymbol{P}^T,\ \mathbf{\Gamma}^T,\ \boldsymbol{l}^T]^T$ and define

$$\mathbb{J}_\Gamma = \begin{pmatrix} \hat{\mathbf{\Pi}} & \hat{\boldsymbol{P}} & \hat{\mathbf{\Gamma}} & \mathbf{0} \\ \hat{\boldsymbol{P}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hat{\mathbf{\Gamma}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}. \tag{4}$$

It is easily verified that

$$\dot{\boldsymbol{\nu}}_\Gamma = \mathbb{J}_\Gamma \nabla H \quad \text{where} \quad H(\boldsymbol{\nu}_\Gamma) = \frac{1}{2}\boldsymbol{\nu} \cdot \mathbb{M}^{-1}\boldsymbol{\nu} - \boldsymbol{r} \cdot mg\mathbf{\Gamma}.$$

The functions $C_{\Gamma_1}(\boldsymbol{\nu}_\Gamma) = \frac{1}{2}\boldsymbol{P} \cdot \boldsymbol{P}$, $C_{\Gamma_2}(\boldsymbol{\nu}_\Gamma) = \frac{1}{2}\mathbf{\Gamma} \cdot \mathbf{\Gamma}$, and $C_{\Gamma_3}(\boldsymbol{\nu}_\Gamma) = \boldsymbol{P} \cdot \mathbf{\Gamma}$ and each component of $\boldsymbol{l}$ are Casimir functions for the uncontrolled system. In fact, C_{Γ_1}, C_{Γ_2}, and C_{Γ_3} are conserved for any choice of control $\boldsymbol{u}$.

2.2 *Viscous Forces*

The model described in Section 2.1 derives from a potential flow description of the fluid dynamics. We append to this model, in the form of generalized forces, additional terms intended to describe the effect of fluid viscosity on the vehicle motion. Traditionally, control design for UV's has relied on local models for hydrodynamic effects, such as lift and drag, which are based on the theory of steady fluid flow and on experimental data. See (Hoerner, 1965, 1975) for more information on hydrodynamic modeling. While such models cannot predict with great accuracy the effect of an unsteady, three-dimensional viscous flow over a maneuvering vehicle, they are effective tools for local control design. Indeed, a principal aim of feedback control is to compensate for necessarily simplistic physical models. With these observations in mind, we seek a simple viscous model which captures the essential nature of drag but imposes few restrictions on the form of the viscous force and moment.

We assume that the viscous force acting on the ellipsoidal vehicle is described by a once continuously differentiable function $\boldsymbol{f_v}(\boldsymbol{\Omega}, \boldsymbol{v})$ where $\boldsymbol{f_v}(\boldsymbol{\Omega}, \boldsymbol{v}) = \boldsymbol{0}$ if and only if $\boldsymbol{v} = \boldsymbol{0}$. Similarly, we assume that the viscous torque acting on the vehicle takes the form $\boldsymbol{f_\Omega}(\boldsymbol{\Omega}, \boldsymbol{v})$ where $\boldsymbol{f_\Omega}(\cdot, \cdot)$ is C^1 and $\boldsymbol{f_\Omega}(\boldsymbol{\Omega}, \boldsymbol{v}) = \boldsymbol{0}$ if and only if $\boldsymbol{\Omega} = \boldsymbol{0}$. The smoothness assumptions are consistent with standard models, which describe fluid force and moment as smooth functions of vehicle velocity; see Etkin (1972), for example. An underlying assumption is that the boundary layer between the vehicle and the free stream does not separate. This assumption is often made for control design purposes, although flow separation can certainly occur on a maneuvering vehicle and can significantly affect the dynamics; see Wetzel & Simpson (1998).

It is well-known that drag opposes velocity in the sense that $\boldsymbol{f_v}(\boldsymbol{\Omega}, \boldsymbol{v}) \cdot \boldsymbol{v} \leq 0$. We make the slightly stronger assumption that the viscous force and moment grow at least linearly with velocity,

$$\begin{aligned} \Omega_i \, \boldsymbol{e_i} \cdot \boldsymbol{f_\Omega}(\boldsymbol{\Omega}, \boldsymbol{v}) &\leq -\underline{f}_{\Omega_i} \Omega_i^2 < 0 \qquad (\Omega_i \neq 0) \\ v_i \, \boldsymbol{e_i} \cdot \boldsymbol{f_v}(\boldsymbol{\Omega}, \boldsymbol{v}) &\leq -\underline{f}_{v_i} v_i^2 < 0 \qquad (v_i \neq 0) \end{aligned} \tag{5}$$

for $i = 1$, 2, and 3, where $\underline{f}_{\Omega_i}$ and $\underline{f}_{v_i}$ are positive scalars. Assumption (5) is consistent, for example, with the linear-plus-quadratic drag model suggested by Fossen (1995).

Because the coordinate axes $\boldsymbol{e_1}$ and $\boldsymbol{e_2}$ describe a plane of symmetry, the vehicle will experience no lift when translating steadily along its 1-axis, i.e., at zero angle of attack (assuming that the flow does not separate). Similarly, because the coordinate axes $\boldsymbol{e_1}$ and $\boldsymbol{e_3}$ describe a plane of symmetry, the vehicle will experience no side force when translating steadily along its 1-axis, i.e., at zero sideslip angle. Therefore, for any scalar c,

$$\boldsymbol{e_i} \cdot \boldsymbol{f_v}(\boldsymbol{0}, c\, \boldsymbol{e_1}) = 0, \qquad \text{for } i = 2 \text{ and } 3. \tag{6}$$

Because this paper concerns stability of steady vehicle translation, a propulsive force is included to counter the drag force at the desired equilibrium. A realistic propulsion model would include thruster dynamics. To simplify the analysis, however, we model thrust as a constant, body-fixed force $-\boldsymbol{f_v}(\boldsymbol{0}, \tilde{v}_1 \boldsymbol{e_1})$, where $\tilde{v}_1$ is the equilibrium speed. This force exactly balances drag when the vehicle moves at the desired equilibrium.

3 Coincident Centers

In this section, we apply the three step procedure outlined in Section 1 to a UV with coincident centers of buoyancy and gravity, that is, with $\boldsymbol{r} = \boldsymbol{0}$.

3.1 *Stabilization*

It was shown by Bloch et al. (1992) that steady intermediate axis rotation of a spacecraft can be stabilized with a single internal rotor by applying feedback which effectively modifies the spacecraft inertia. Here, we define feedback for a UV with three internal rotors which modifies the inertia and stabilizes steady, long-axis translation. We show, for the conservative model, that the equilibrium can be stabilized using only two rotors. A third rotor is necessary when damping is present.

With $\boldsymbol{r} = \boldsymbol{0}$, and neglecting viscous forces, the equations of motion for a UV with internal rotors are given by (1) with $\boldsymbol{\mathcal{T}}$ and $\boldsymbol{\mathcal{F}}$ zero. If, in addition, $\boldsymbol{u}$ is chosen to maintain the rotors *locked in place*, then the first two equations of (1) describe a rigid ellipsoid in a perfect fluid, as considered by Holmes et al. (1998). This system exhibits rich dynamics, with several families of equilibria. There are three two-parameter families of "pure mode" equilibria corresponding to steady translation along and rotation about vehicle principal axes. Depending on the ordering of moments of inertia Λ_1, Λ_2, and Λ_3, there may also exist non-principal axis steady motions referred to as "mixed mode" equilibria.

The pure 1 mode equilibrium

$$\boldsymbol{\Pi}_e = \Pi_1^0 \boldsymbol{e_1}, \quad \boldsymbol{P}_e = P_1^0 \boldsymbol{e_1}, \quad (\boldsymbol{l}_e = \boldsymbol{J_r} \boldsymbol{\Lambda}^{-1} \boldsymbol{\Pi}_e), \tag{7}$$

is of practical interest because it corresponds to a streamlined motion. The subscript "e" in (7) denotes an equilibrium value of a momentum vector and the superscript "0" denotes an equilibrium value of a momentum component. Holmes et al. (1998) showed that (7) is stable provided

$$\frac{1}{\Lambda_1}\left(\frac{1}{\Lambda_i}-\frac{1}{\Lambda_1}\right)\left(\frac{\Pi_1^0}{P_1^0}\right)^2 > \frac{1}{\Lambda_i}\left(\frac{1}{m_1}-\frac{1}{m_i}\right) \tag{8}$$

for both $i = 2$ and $i = 3$. Given that $L_1 > L_2 > L_3$, the moments of inertia may be ordered $\Lambda_3 > \Lambda_2 > \Lambda_1$, $\Lambda_2 > \Lambda_3 > \Lambda_1$, or $\Lambda_2 > \Lambda_1 > \Lambda_3$, depending on the ratios $L_1 : L_2 : L_3$. In any case, the right-hand side of (8) will be positive and the left-hand side will be negative for $i = 2$. The sufficient conditions for stability cannot be satisfied. In fact, it is well known that steady, long-axis translation of a non-rotating rigid ellipsoid through a fluid is unstable.

To overcome this problem, we design a feedback control law for the internal rotors which stabilizes the equilibrium (7). With the goal of obtaining a closed-loop Hamiltonian system with control-modified inertia, and following in the spirit of (Bloch et al., 1992), we define the feedback control law

$$\boldsymbol{u} = \boldsymbol{K}\dot{\boldsymbol{\Pi}} = \boldsymbol{K}\big(\boldsymbol{\Pi}\times\boldsymbol{\Omega} + \boldsymbol{P}\times\boldsymbol{v}\big), \tag{9}$$

where $\boldsymbol{K} = \mathrm{diag}(k_1, k_2, k_3)$ is a matrix of control gains. This control law can be derived as an application of the method of controlled Lagrangians; see (Woolsey, 2001).

Since $\dot{\boldsymbol{l}} = \boldsymbol{u} = \boldsymbol{K}\dot{\boldsymbol{\Pi}}$, the vector quantity $\boldsymbol{l} - \boldsymbol{K}\boldsymbol{\Pi}$ is conserved. It is convenient to change variables from $(\boldsymbol{\Pi}, \boldsymbol{P}, \boldsymbol{l})$ to $(\boldsymbol{\Pi}, \boldsymbol{P}, \boldsymbol{\zeta})$, where

$$\boldsymbol{\zeta} = (\mathbb{I} - \boldsymbol{K})^{-1}(\boldsymbol{l} - \boldsymbol{K}\boldsymbol{\Pi}). \tag{10}$$

(We anticipate choosing $k_i \neq 1$ for $i = 1, 2$, and 3 so that $\boldsymbol{\zeta}$ is well-defined.) In these new coordinates, the closed-loop equations of motion are

$$\begin{aligned}\dot{\boldsymbol{\Pi}} &= \boldsymbol{\Pi}\times\boldsymbol{\Omega} + \boldsymbol{P}\times\boldsymbol{v}\\ \dot{\boldsymbol{P}} &= \boldsymbol{P}\times\boldsymbol{\Omega}\\ \dot{\boldsymbol{\zeta}} &= \boldsymbol{0}.\end{aligned} \tag{11}$$

As intended, equations (11) form a Lie-Poisson system. To see this, first define the "controlled inertia matrix"

$$\boldsymbol{I_K} = \mathrm{diag}(I_{K_1}, I_{K_2}, I_{K_3}) = (\mathbb{I} - \boldsymbol{K})^{-1}(\boldsymbol{\Lambda} - \boldsymbol{J_r}). \tag{12}$$

and note that $\boldsymbol{\Omega} = \boldsymbol{I_K}^{-1}(\boldsymbol{\Pi} - \boldsymbol{\zeta})$. Defining a new momentum vector $\tilde{\boldsymbol{\nu}} = [\boldsymbol{\Pi}^T,\ \boldsymbol{P}^T,\ \boldsymbol{\zeta}^T]^T$, equations (11) may be written

$$\dot{\tilde{\boldsymbol{\nu}}} = \mathbb{J}\nabla H_K$$

where $\mathbb{J}$ is given by (2) and

$$H_K = \frac{1}{2}(\boldsymbol{\Pi} - \boldsymbol{\zeta})\cdot\boldsymbol{I_K}^{-1}(\boldsymbol{\Pi} - \boldsymbol{\zeta}) + \frac{1}{2}\boldsymbol{P}\cdot\boldsymbol{M}^{-1}\boldsymbol{P}. \tag{13}$$

Notice that $\boldsymbol{I_K}$ plays the role of inertia for the closed-loop system. The new Hamiltonian H_K is a control-dependent modification of the kinetic energy of the uncontrolled system. The functions $C_1 = \frac{1}{2}\boldsymbol{P}\cdot\boldsymbol{P}$ and $C_2 = \boldsymbol{\Pi}\cdot\boldsymbol{P}$, as well as each component of $\boldsymbol{\zeta}$, are five Casimir functions for this Lie-Poisson system.

Pure mode equilibria of the uncontrolled system, such as the equilibrium (7), are also equilibria of the closed-loop system (11) with $\boldsymbol{\zeta}_e \parallel \boldsymbol{\Pi}_e \parallel \boldsymbol{P}_e$. We focus on the three-parameter family of pure 1 mode equilibria

$$\boldsymbol{\Pi}_e = \Pi_1^0\boldsymbol{e_1} \quad \boldsymbol{P}_e = P_1^0\boldsymbol{e_1} \quad \boldsymbol{\zeta}_e = \zeta_1^0\boldsymbol{e_1}. \tag{14}$$

These equilibria correspond to vehicle translation along and rotation about the 1-axis with the 1-axis rotor spinning at a constant rate.

Theorem 1 ***(Lyapunov Stability)*** *Let* $\mathrm{sign}(I_{K_2}) = \mathrm{sign}(I_{K_3})$. *Then, the equilibrium (14) with* $P_1^0 \neq 0$ *is stable if, for* $i = 2$ *and* $i = 3$,

$$\begin{aligned}&\frac{1}{I_{K_1}}\left(\frac{1}{I_{K_i}}-\frac{1}{I_{K_1}}\right)\left(\frac{\Pi_1^0-\zeta_1^0}{P_1^0}\right)^2\\ &+\frac{1}{I_{K_1}I_{K_i}}\frac{(\Pi_1^0-\zeta_1^0)\zeta_1^0}{(P_1^0)^2} > \frac{1}{I_{K_i}}\left(\frac{1}{m_1}-\frac{1}{m_i}\right).\end{aligned} \tag{15}$$

The proof is an application of the energy-Casimir method (Marsden & Ratiu, 1994). Briefly, one defines

$$H_\Phi = H_K(\boldsymbol{\Pi}, \boldsymbol{P}, \boldsymbol{\zeta}) + \Phi(C_1, C_2, \zeta_1, \zeta_2, \zeta_3), \tag{16}$$

and seeks conditions on the function Φ under which H_Φ has a minimum or a maximum at the equilibrium (14). Doing so, one obtains a function H_Φ for which the equilibrium is a maximum. See (Woolsey, 2001) for details.

When $\zeta_1^0 = 0$, conditions (15) revert to conditions (8) with $\boldsymbol{\Lambda}$ replaced by $\boldsymbol{I_K}$. Of course, since $\boldsymbol{I_K}$ is parameterized by the control, the conditions (15) can be satisfied by a suitable choice of k_1, k_2, and k_3.

Practically, the case where $\zeta_1^0 = 0$ is less interesting than the case where $\Omega_1^0 = (\Pi_1^0 - \zeta_1^0)/I_{K_1} = 0$, i.e., the case where the vehicle does not spin. If $\Omega_1^0 = 0$, conditions (15) become

$$0 > \frac{1}{I_{K_i}}\left(\frac{1}{m_1}-\frac{1}{m_i}\right) \qquad i = 2 \text{ and } 3. \tag{17}$$

Since $m_1 < m_2 < m_3$, conditions (17) will hold only when $I_{K_2} < 0$ and $I_{K_3} < 0$.

Corollary 2 *Choosing* $k_2 > 1$ *and* $k_3 > 1$ *stabilizes steady, long-axis translation with zero body angular rate.*

The stability conditions (17) do not involve I_{K_1}. These conditions hold even if $k_1 = 0$ and $I_{K_1} = \Lambda_1 = I_1$. Thus, for the conservative model, steady, long-axis translation may be stabilized using only two internal rotors.

Remark 3 *If one also chooses $k_1 > 1$, then a Lyapunov function for (14) with $\Pi_1^0 = \zeta_1^0$ (i.e., with $\Omega_1^0 = 0$) is*

$$\begin{aligned}\bar{H}_\Phi &= H_K - \frac{1}{m_1}C_1 + \frac{1}{2}\rho_{11}(C_1 - \frac{1}{2}(P_1^0)^2)^2 \\ &+ \rho_{12}(C_2 - \Pi_1^0 P_1^0)(C_1 - \frac{1}{2}(P_1^0)^2) + \frac{1}{2}\rho_{22}(C_2 - \Pi_1^0 P_1^0)^2 \\ &+ \frac{1}{2}\rho_{44}\zeta_2^2 + \frac{1}{2}\rho_{55}\zeta_3^2, \qquad (18)\end{aligned}$$

where $\rho_{44} < 0$, $\rho_{55} < 0$, and the remaining constants ρ_{11}, ρ_{12}, and ρ_{22} are chosen to make $\bar{H}_\Phi$ negative definite at the equilibrium (Woolsey, 2001). The function (18) is relevant in later sections where, for reasons concerning the effect of physical dissipation, we choose $\boldsymbol{I_K} < \boldsymbol{0}$.

3.2 Asymptotic Stabilization

Next, we design feedback dissipation to asymptotically stabilize long-axis translation without vehicle rotation,

$$\boldsymbol{\Pi}_e = \Pi_1^0 \boldsymbol{e_1}, \quad \boldsymbol{P}_e = P_1^0 \boldsymbol{e_1}, \quad \boldsymbol{\zeta}_e = \Pi_1^0 \boldsymbol{e_1}. \qquad (19)$$

The Lyapunov function (18) is used to construct a dissipative feedback control law which asymptotically stabilizes (19).

Referring to equations (1), with $\boldsymbol{\mathcal{T}}$ and $\boldsymbol{\mathcal{F}}$ zero, let

$$\boldsymbol{u} = \boldsymbol{K}\left(\boldsymbol{\Pi} \times \boldsymbol{\Omega} + \boldsymbol{P} \times \boldsymbol{v}\right) + (\mathbb{I} - \boldsymbol{K})\boldsymbol{u_d} \qquad (20)$$

where $(\mathbb{I} - \boldsymbol{K}) < \boldsymbol{0}$ and where $\boldsymbol{u_d}$ represents a dissipative control term which remains to be chosen. Again, make the change of variables $(\boldsymbol{\Pi}, \boldsymbol{P}, \boldsymbol{l}) \to (\boldsymbol{\Pi}, \boldsymbol{P}, \boldsymbol{\zeta})$ defined by (10). The equations of motion become

$$\begin{aligned}\dot{\boldsymbol{\Pi}} &= \boldsymbol{\Pi} \times \boldsymbol{\Omega} + \boldsymbol{P} \times \boldsymbol{v} \\ \dot{\boldsymbol{P}} &= \boldsymbol{P} \times \boldsymbol{\Omega} \\ \dot{\boldsymbol{\zeta}} &= \boldsymbol{u_d}. \qquad (21)\end{aligned}$$

Recall that the equilibrium (19) is a maximum of $\bar{H}_\Phi$ given by equation (18). To make $\dot{\bar{H}}_\Phi \geq 0$, we note that $\dot{\bar{H}}_\Phi = \left(\frac{\partial}{\partial \boldsymbol{\zeta}}\bar{H}_\Phi\right) \cdot \boldsymbol{u_d}$, and we let

$$\boldsymbol{u_d} = \boldsymbol{K_d}\frac{\partial \bar{H}_\Phi}{\partial \zeta} = \boldsymbol{K_d}\left(-\boldsymbol{\Omega} + \begin{pmatrix} 0 \\ \rho_{44}\zeta_2 \\ \rho_{55}\zeta_3 \end{pmatrix}\right) \qquad (22)$$

where $\boldsymbol{K_d} > \boldsymbol{0}$ is a matrix of control gains.

Noting that $C_1 = \frac{1}{2}\boldsymbol{P} \cdot \boldsymbol{P}$ and $C_2 = \boldsymbol{\Pi} \cdot \boldsymbol{P}$ are conserved for equations (21) (regardless of the choice of $\boldsymbol{u_d}$), let $\mathcal{D}_\Phi = \{\tilde{\boldsymbol{\nu}} \mid \frac{1}{2}\boldsymbol{P} \cdot \boldsymbol{P} = C_1, \boldsymbol{\Pi} \cdot \boldsymbol{P} = C_2\}$ represent the invariant subspace on which the dynamics (21) evolve. If ω_Φ is some compact, positively invariant subset of $\mathcal{D}_\Phi$ then, by LaSalle's invariance principle, solutions starting in ω_Φ converge to the largest invariant set $\mathcal{M}$ contained in the set $E = \{\tilde{\boldsymbol{\nu}} \in \omega_\Phi \mid \dot{\bar{H}}_\Phi = 0\}$.

One can show that, for any $\omega_\Phi \subset \mathcal{D}_\Phi$, the set $\mathcal{M}$ contains only equilibria (Woolsey, 2001). We choose ω_Φ to exclude undesired equilibria. Let

$$\omega = \left\{\tilde{\boldsymbol{\nu}} \in \mathcal{D}_\Phi \;\middle|\; \bar{H}_\Phi \geq (1 - \epsilon)\left(\frac{1}{m_2} - \frac{1}{m_1}\right)C_1\right\}$$

where $0 < \epsilon \ll 1$. The set ω is positively invariant, because $\bar{H}_\Phi$ is nondecreasing, and it excludes states for which $P_1 = 0$. Let $\omega_\Phi = \{\tilde{\boldsymbol{\nu}} \in \omega \mid P_1 > 0\}$. If one chooses $k_2 > 1$, $k_3 > 1$, $\rho_{44} < 0$ and $\rho_{55} < 0$ to also satisfy

$$I_{K_2} + \frac{1}{\rho_{44}} < I_{K_3} + \frac{1}{\rho_{55}}, \qquad (23)$$

then ω_Φ contains no mixed-mode equilibria. In fact, the only equilibrium contained in ω_Φ is (19) with $P_1^0 > 0$.

Theorem 4 ***(Asymptotic Stability)*** *Suppose that $C_1 \neq 0$. Then any solution of equations (21), with $\boldsymbol{u_d}$ given by (22), which starts in ω_Φ converges to*

$$\boldsymbol{\Pi}_e = \frac{C_2}{\sqrt{2C_1}}\,\boldsymbol{e_1}, \;\; \boldsymbol{P}_e = \sqrt{2C_1}\,\boldsymbol{e_1}, \;\; \boldsymbol{\zeta}_e = \frac{C_2}{\sqrt{2C_1}}\,\boldsymbol{e_1}. \qquad (24)$$

The proof of Theorem 4 is an application of LaSalle's invariance principle. The theorem implies that the body angular velocity goes to zero as do the angular velocities of the 2-axis and 3-axis internal rotors. The vehicle motion converges to pure translation along the long axis with the 1-axis internal rotor spinning at some generically nonzero rate. The magnitudes of $\boldsymbol{\Pi}_e$, $\boldsymbol{P}_e$, and $\boldsymbol{\zeta}_e$ are determined by the conservation laws. Simulations verify that ω_Φ is a quite conservative estimate for the region of attraction of (19). Moreover, further analysis suggests conditions on control parameters to increase the size of ω_Φ. See (Woolsey, 2001).

3.3 Including Viscous Forces

In this section, we apply the control law (20) to a UV model which includes viscous forces and torques. The open-loop equations are given by (1) with

$$\begin{aligned}\boldsymbol{\mathcal{T}} &= \boldsymbol{f_\Omega}(\boldsymbol{\Omega}, \boldsymbol{v}) \\ \boldsymbol{\mathcal{F}} &= \boldsymbol{f_v}(\boldsymbol{\Omega}, \boldsymbol{v}) - \boldsymbol{f_v}(\boldsymbol{0}, \tilde{v}_1\boldsymbol{e_1}),\end{aligned}$$

where $\tilde{v}_1 > 0$ is the desired speed. A crucial requirement is that thrust and drag equilibrate when $\boldsymbol{\Omega} \equiv \boldsymbol{0}$ and

$v_2 \equiv v_3 \equiv 0$ in such a way that $v_1 \to \tilde{v}_1$. This requirement leads to the following restriction on $\tilde{v}_1$:

$$(v_1 - \tilde{v}_1)\boldsymbol{e_1} \cdot (\boldsymbol{f_v}(\boldsymbol{0}, v_1\boldsymbol{e_1}) - \boldsymbol{f_v}(\boldsymbol{0}, \tilde{v}_1\boldsymbol{e_1})) < 0 \tag{25}$$

when $v_1 \neq \tilde{v}_1$. Assumption (25) requires that, when the vehicle translates along its long axis at a speed faster (slower) than $\tilde{v}_1$, the magnitude of drag is larger (smaller) than the magnitude of thrust. This is not a restrictive assumption, although one might expect a small range of inadmissible equilibrium speeds in the neighborhood of the critical speed for boundary layer transition, where the drag force can *decrease* with increasing speed (Hoerner, 1965).

For the conservative system, the control law provided by the method of controlled Lagrangians is

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{K}\dot{\boldsymbol{\Pi}} && (26)\\ &= \boldsymbol{K}(\boldsymbol{\Pi} \times \boldsymbol{\Omega} + \boldsymbol{P} \times \boldsymbol{v}). && (27) \end{aligned}$$

When drag and thrust are included, (26) becomes

$$\boldsymbol{u} = \boldsymbol{K}(\boldsymbol{\Pi} \times \boldsymbol{\Omega} + \boldsymbol{P} \times \boldsymbol{v} + \boldsymbol{f_\Omega}(\boldsymbol{\Omega}, \boldsymbol{v})). \tag{28}$$

This approach was taken in (Woolsey & Leonard, 1999b), where it was shown that the desired equilibrium of the resulting closed-loop system is destabilized by drag, as one might expect for an equilibrium which is an "energy maximum." The detrimental effect of drag was overcome by applying appropriate feedback dissipation.

Here, we apply the original feedback control law (27), as defined for the conservative system model, instead of (28). The desired closed-loop equilibrium is still a maximum of the function $\bar{H}_\Phi$ given by (18), however drag tends to increase the modified energy, driving the state to the desired equilibrium asymptotically. In fact, one can view the compensatory feedback dissipation formulated in (Woolsey & Leonard, 1999b) as undoing the harm done by choosing the control law (28). (See Remark 6 at the end of this section.) Note that the control law (27) requires neither acceleration measurements nor a model of physical damping.

The construction of the Lyapunov function $\bar{H}_\Phi$ in Section 3.1 relied on several conservation laws which are broken when physical dissipation is introduced. As a result, $\dot{\bar{H}}_\Phi$ is indefinite when viscous forces are included. Thus, $\bar{H}_\Phi$ can no longer serve as a Lyapunov function. In this section, a *semidefinite* Lyapunov function is formed by dropping those terms of $\bar{H}_\Phi$ which destroy the definiteness of $\dot{\bar{H}}_\Phi$. This semidefinite function allows one to characterize the vehicle dynamics, leading to a global asymptotic stability result.

Once again, choose $\boldsymbol{u}$ as in (20) and make the change of variables $(\boldsymbol{\Pi}, \boldsymbol{P}, \boldsymbol{l}) \to (\boldsymbol{\Pi}, \boldsymbol{P}, \boldsymbol{\zeta})$ defined by (10). The closed-loop equations of motion are

$$\begin{aligned} \dot{\boldsymbol{\Pi}} &= \boldsymbol{\Pi} \times \boldsymbol{\Omega} + \boldsymbol{P} \times \boldsymbol{v} + \boldsymbol{f_\Omega}(\boldsymbol{\Omega}, \boldsymbol{v}) \\ \dot{\boldsymbol{P}} &= \boldsymbol{P} \times \boldsymbol{\Omega} + \boldsymbol{f_v}(\boldsymbol{\Omega}, \boldsymbol{v}) - \boldsymbol{f_v}(\boldsymbol{0}, \boldsymbol{v}_e) \\ \dot{\boldsymbol{\zeta}} &= -(\mathbb{I} - \boldsymbol{K})^{-1}\boldsymbol{K}\boldsymbol{f_\Omega}(\boldsymbol{\Omega}, \boldsymbol{v}) + \boldsymbol{u_d}. \end{aligned} \tag{29}$$

Because the terms in $\bar{H}_\Phi$ (18) which are quadratic in the Casimirs make the rate $\dot{\bar{H}}_\Phi$ indefinite when drag is introduced, we omit these terms to obtain the negative semidefinite function

$$V = \frac{1}{2}\left((\boldsymbol{\Pi} - \boldsymbol{\zeta}) \cdot \boldsymbol{I_K}^{-1}(\boldsymbol{\Pi} - \boldsymbol{\zeta}) + \boldsymbol{P} \cdot (\boldsymbol{M}^{-1} - \frac{1}{m_1}\mathbb{I})\boldsymbol{P}\right).$$

The desired equilibrium maximizes V, but it is not a *strict* maximum. In terms of body velocities,

$$V = \frac{1}{2}\boldsymbol{\Omega} \cdot \boldsymbol{I_K}\boldsymbol{\Omega} + \sum_{i=1}^{2} \frac{1}{2}\left(\frac{1}{m_i} - \frac{1}{m_1}\right)(m_i v_i)^2,$$

so V is maximum whenever $\boldsymbol{\Omega} = \boldsymbol{0}$ and $v_2 = v_3 = 0$, but V is independent of v_1 and $\boldsymbol{\Omega_r}$. Differentiating V gives

$$\begin{aligned} \dot{V} = \boldsymbol{\Omega} \cdot &\left((\mathbb{I} - \boldsymbol{K})^{-1}\boldsymbol{f_\Omega}(\boldsymbol{\Omega}, \boldsymbol{v}) - \boldsymbol{u_d}\right) \\ &+ \sum_{i=2}^{3} \frac{(m_1 - m_i)}{m_1} v_i \, \boldsymbol{e_i} \cdot (\boldsymbol{f_v}(\boldsymbol{\Omega}, \boldsymbol{v}) - \boldsymbol{f_v}(\boldsymbol{0}, \boldsymbol{v}_e)). \end{aligned}$$

Define the dissipative feedback

$$\boldsymbol{u_d} = -\boldsymbol{K_d}\boldsymbol{\Omega} \tag{30}$$

where $\boldsymbol{K_d} = \mathrm{diag}(k_{d_1},\ k_{d_2},\ k_{d_3}) \geq \boldsymbol{0}$. Under assumptions (5) and (6) on the form of drag, $\dot{V} \geq 0$. Since V is bounded above and nondecreasing, $\dot{V} \to 0$ asymptotically. Consequently, $\boldsymbol{\Omega}$, v_2, and v_3 converge to zero. Because of (25), one may further conclude that v_1 goes asymptotically to the desired speed $\tilde{v}_1$. In fact, V converges to zero exponentially, allowing one to show that $\|\boldsymbol{\Omega_r}\|$ is bounded and converges to a constant value (Woolsey, 2001).

Theorem 5 ***(Global Asymptotic Stability)*** *Equations (29) and (30) with* $(\mathbb{I} - \boldsymbol{K}) < \boldsymbol{0}$ *and* $\boldsymbol{K_d} \geq 0$ *describe a system whose state remains bounded and converges to an equilibrium*

$$\boldsymbol{\Pi_e} = \tilde{\boldsymbol{\Pi}}, \quad \boldsymbol{P_e} = m_1\tilde{v}_1\boldsymbol{e_1}, \quad \boldsymbol{\zeta}_e = \tilde{\boldsymbol{\Pi}}. \tag{31}$$

The vehicle's translational and angular velocity will always approach the desired values $\boldsymbol{v}_e = \tilde{v}_1\boldsymbol{e_1}$ *and* $\boldsymbol{\Omega}_e = \boldsymbol{0}$*, although the final equilibrium value of* $\boldsymbol{\Pi}$ *and* $\boldsymbol{\zeta}$ *will vary with initial condition.*

Remark 6 *The control law $\boldsymbol{u} = \boldsymbol{K}\dot{\boldsymbol{\Pi}}$ given in (26), with $\mathbb{I} - \boldsymbol{K} < \boldsymbol{0}$, effectively changes the sign of the vehicle inertia. Physically, this means that external torques will tend to turn the vehicle in the opposite sense, as compared to the uncontrolled setting. The effect of any torque which would drive the uncontrolled vehicle away from the desired (unstable) equilibrium is reversed by the feedback control law. In particular, the effect of the destabilizing term $\boldsymbol{P} \times \boldsymbol{v}$ which appears in (1) is reversed. If $\dot{\boldsymbol{\Pi}}$ includes a drag term, as in (28), the feedback control law reverses its effect as well. If the viscous torque is simply excluded from the feedback control law, as in (27), its effect will not be reversed and drag will tend to decrease the body angular velocity.*

4 Noncoincident Centers

This section treats the more general case of a UV whose CG lies along the shortest ellipsoid principal axis. We assume that $\boldsymbol{r} = \gamma \boldsymbol{e_3}$ with γ a scalar constant. Steady long-axis translation is an equilibrium of practical interest which is unstable for the uncontrolled system (Leonard, 1997a). Even though a low CG ($\gamma > 0$) provides a restoring torque in pitch and roll, the fluid causes the vehicle to yaw away from the equilibrium.

Neglecting viscous forces, the dynamics for a UV with noncoincident CB and CG are described by equations (3), with $\boldsymbol{\mathcal{T}}$ and $\boldsymbol{\mathcal{F}}$ zero. As described in Section 2.1, when $\boldsymbol{u} = \boldsymbol{0}$, the dynamics are Lie-Poisson where the Hamiltonian is kinetic plus potential energy. Applying the method of controlled Lagrangians suggests choosing

$$\begin{aligned}\boldsymbol{u} &= \boldsymbol{K}\left(\dot{\boldsymbol{\Pi}} - (m\hat{\boldsymbol{r}})\boldsymbol{M}^{-1}\dot{\boldsymbol{P}}\right)\\ &= \boldsymbol{K}\Big((\boldsymbol{\Pi} \times \boldsymbol{\Omega} + \boldsymbol{P} \times \boldsymbol{v} + \boldsymbol{r} \times mg\boldsymbol{\Gamma})\\ &\qquad -(m\hat{\boldsymbol{r}})\boldsymbol{M}^{-1}(\boldsymbol{P} \times \boldsymbol{\Omega})\Big)\end{aligned} \tag{32}$$

where $\boldsymbol{K} = \operatorname{diag}(k_1, k_2, k_3)$ is a control gain matrix (Woolsey, 2001). When $\boldsymbol{r} = \boldsymbol{0}$, (32) reduces to the control law (9) chosen in Section 3.1.

As in Section 3, the control law (32) effectively modifies the kinetic energy metric. Define the square matrices $\boldsymbol{A}, \boldsymbol{B}$, and $\boldsymbol{C}$ as

$$\begin{pmatrix} \boldsymbol{A} & \boldsymbol{B}^T \\ \boldsymbol{B} & \boldsymbol{C} \end{pmatrix} = \begin{pmatrix} (\boldsymbol{\Lambda} - \boldsymbol{J_r}) & m\hat{\boldsymbol{r}} \\ -m\hat{\boldsymbol{r}} & \boldsymbol{M} \end{pmatrix}^{-1}$$

and define the control-modified components

$$\begin{aligned}\boldsymbol{A_K} &= \boldsymbol{A}(\mathbb{I} - \boldsymbol{K})\\ \boldsymbol{B_K} &= \boldsymbol{M}^{-1}(m\hat{\boldsymbol{r}})\boldsymbol{A_K}\\ \boldsymbol{C_K} &= \boldsymbol{C}(\boldsymbol{M} + (m\hat{\boldsymbol{r}})(\boldsymbol{\Lambda} - \boldsymbol{J_r})^{-1}\boldsymbol{K}(m\hat{\boldsymbol{r}}))\boldsymbol{M}^{-1}.\end{aligned} \tag{33}$$

We change variables from $(\boldsymbol{\Pi}, \boldsymbol{P}, \boldsymbol{\Gamma}, \boldsymbol{l})$ to $(\boldsymbol{\Pi}, \boldsymbol{P}, \boldsymbol{\Gamma}, \boldsymbol{\zeta})$, where

$$\boldsymbol{\zeta} = (\mathbb{I} - \boldsymbol{K})^{-1}\left(\boldsymbol{l} - \boldsymbol{K}(\boldsymbol{\Pi} - m\hat{\boldsymbol{r}}\boldsymbol{M}^{-1}\boldsymbol{P})\right). \tag{34}$$

This definition of $\boldsymbol{\zeta}$ reduces to the previous definition (10) when $\boldsymbol{r} = \boldsymbol{0}$. Let $\tilde{\boldsymbol{\nu}}_\Gamma = [\boldsymbol{\Pi}^T,\ \boldsymbol{P}^T,\ \boldsymbol{\Gamma}^T,\ \boldsymbol{\zeta}^T]^T$. The closed-loop equations of motion are Lie-Poisson:

$$\dot{\tilde{\boldsymbol{\nu}}}_\Gamma = \mathbb{J}_\Gamma \nabla H_K,$$

where $\mathbb{J}_\Gamma$ is given by (4) and

$$\begin{aligned}H_K = &\frac{1}{2}(\boldsymbol{\Pi} - \boldsymbol{\zeta}) \cdot \boldsymbol{A_K}(\boldsymbol{\Pi} - \boldsymbol{\zeta}) + \boldsymbol{P} \cdot \boldsymbol{B_K}(\boldsymbol{\Pi} - \boldsymbol{\zeta})\\ &+\frac{1}{2}\boldsymbol{P} \cdot \boldsymbol{C_K}\boldsymbol{P} - \boldsymbol{r} \cdot mg\boldsymbol{\Gamma}.\end{aligned} \tag{35}$$

This Hamiltonian reduces to the previous definition (13) when $\boldsymbol{r} = \boldsymbol{0}$. There are six independent Casimir functions, $C_{\Gamma_1} = \frac{1}{2}\boldsymbol{P} \cdot \boldsymbol{P}$, $C_{\Gamma_2} = \frac{1}{2}\boldsymbol{\Gamma} \cdot \boldsymbol{\Gamma}$, $C_{\Gamma_3} = \boldsymbol{P} \cdot \boldsymbol{\Gamma}$, and each component of $\boldsymbol{\zeta}$. The desired equilibrium is pure 1-axis translation in the horizontal plane, so we consider the special case $C_{\Gamma_3} = 0$. (The Casimir C_{Γ_3} is conserved regardless of the control law. Thus, we consider the dynamics restricted to a particular invariant subspace.)

We anticipate a dissipative feedback control law which drives the system to a translational equilibrium, that is, an equilibrium for which $\boldsymbol{\Omega}$ is zero. With $\boldsymbol{r} = \gamma\boldsymbol{e_3}$, there may exist non-principal axis translational equilibria (Woolsey, 2001). However, choosing γ such that

$$\gamma^2 \geq \left(\frac{2C_{\Gamma_1}}{mg}\right)^2 \left(\frac{1}{m_1} - \frac{1}{m_3}\right)^2 \tag{36}$$

ensures that the only equilibria for which $\boldsymbol{\Omega}$ is zero are pure principal-axis translational equilibria. Of interest here is the family of equilibria for which the vehicle translates along its long axis in the horizontal plane,

$$\begin{aligned}&\boldsymbol{\Pi}_e = \frac{m\gamma}{m_1}\sqrt{2C_{\Gamma_1}}\boldsymbol{e_2} + \boldsymbol{\zeta}_e, \qquad \boldsymbol{P}_e = \sqrt{2C_{\Gamma_1}}\boldsymbol{e_1},\\ &\boldsymbol{\Gamma}_e = \boldsymbol{e_3}, \qquad \boldsymbol{\zeta}_e = [\zeta_1^0,\ \zeta_2^0,\ \zeta_3^0]^T.\end{aligned} \tag{37}$$

Theorem 7 *The equilibrium (37) with $C_{\Gamma_1} \neq 0$ is stable if $k_i > 1$ ($i = 1, 2, 3$) and $\gamma < 0$ satisfies (36).*

The proof is an application of the energy-Casimir method. Interestingly, the theorem requires that the CG be *above* the CB (i.e., that $\gamma < 0$). This is counterintuitive; one would ordinarily expect the CG to be *below* the CB for stability. For example, Leonard (1997a) showed that steady translation of the vehicle along its intermediate axis in the horizontal plane is stable without control so long as $\gamma > 0$. That stability under the given control law might require a relatively high CG implies the internal rotors will be "balancing the inverted vehicle" as well as stabilizing steady translation. Intuitively, this seems rather inefficient and impractical.

Remark 8 *Since the control $\boldsymbol{u}$ represents a torque applied to the internal rotors, a reaction torque $-\boldsymbol{u}$ acts on the vehicle (less the rotors). The control law (32) with $k_i > 1$ tends to reverse the effect of torques acting on the vehicle. In particular, the destabilizing effect of the term $\boldsymbol{P} \times \boldsymbol{v}$ is reversed. (See Remark 6.) However, the effect of gravity (i.e., the torque $\boldsymbol{r} \times mg\boldsymbol{\Gamma}$) is also reversed, even though gravity ordinarily plays a useful role when the CG is below the CB.*

Rather than pursue the control law (32), we consider the following modified version,

$$\begin{aligned}\boldsymbol{u} = k\left((\boldsymbol{\Pi} \times \boldsymbol{\Omega} + \boldsymbol{P} \times \boldsymbol{v}) - (m\hat{\boldsymbol{r}})\boldsymbol{M}^{-1}(\boldsymbol{P} \times \boldsymbol{\Omega})\right) \\ +(1-k)(\boldsymbol{\zeta} \times \boldsymbol{\Omega} - \tilde{\boldsymbol{u}})\end{aligned} \tag{38}$$

where k is a scalar gain and $\tilde{\boldsymbol{u}}$ represents a dissipative feedback control law to be determined. The control law (38) is a modification of (32) that does not involve the gravitational torque $\boldsymbol{r} \times mg\boldsymbol{\Gamma}$.

Remark 9 *Let $\tilde{\boldsymbol{u}} = \boldsymbol{0}$. The closed-loop system under control law (38) is "almost Poisson" (Cannas da Silva & Weinstein, 1999):*

$$\dot{\tilde{\boldsymbol{\nu}}}_\Gamma = \begin{pmatrix} \hat{\boldsymbol{\Pi}} & \hat{\boldsymbol{P}} & (1-k)\hat{\boldsymbol{\Gamma}} & \boldsymbol{0} \\ \hat{\boldsymbol{P}} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ (1-k)\hat{\boldsymbol{\Gamma}} & \boldsymbol{0} & \boldsymbol{0} & -k\hat{\boldsymbol{\Gamma}} \\ \boldsymbol{0} & \boldsymbol{0} & -k\hat{\boldsymbol{\Gamma}} & -\boldsymbol{\zeta} \end{pmatrix} \nabla H_k, \tag{39}$$

where the new Hamiltonian H_k is

$$\begin{aligned}H_k = \frac{1}{2}(\boldsymbol{\Pi} - \boldsymbol{\zeta}) \cdot \boldsymbol{A_K}(\boldsymbol{\Pi} - \boldsymbol{\zeta}) + \boldsymbol{P} \cdot \boldsymbol{B_K}(\boldsymbol{\Pi} - \boldsymbol{\zeta}) \\ +\frac{1}{2}\boldsymbol{P} \cdot \boldsymbol{C_K}\boldsymbol{P} - \left(\frac{1}{1-k}\right)\boldsymbol{r} \cdot mg\boldsymbol{\Gamma}.\end{aligned}$$

The control parameter k appears in both the kinetic and potential energy terms as well as in the skew-symmetric tensor in equation (39). One may view the control as shaping the potential energy and the dynamic structure as well as the kinetic energy. The system (39) generalizes the system described in (Woolsey & Leonard, 2000) for a vehicle with coincident CB and CG. Potential shaping for UV's is discussed in (Leonard, 1997b). Feedback modification of dynamic structure is discussed in (Blankenstein et al., 2002) and in (Chang et al., 2002).

Theorem 10 *Consider the control law (38), with $k > 1$, and let $\tilde{\boldsymbol{u}} = \boldsymbol{K_d}\boldsymbol{\Omega}$ where $\boldsymbol{K_d} = \mathrm{diag}(k_{d_1}, k_{d_2}, k_{d_3}) > \boldsymbol{0}$. If $\gamma > 0$, then the there is a range of dissipative control gains $k_{d_i} > 0$ and a neighborhood of initial states for which the system converges to an equilibrium of the form (37).*

See (Woolsey, 2001) for a proof of the theorem. Note that this choice of control requires the CG to be below the CB ($\gamma > 0$). The control law thus preserves the naturally stabilizing effect of a low CG.

We note that viscous forces can be treated in the manner of Section 3.3. One obtains a slightly modified version of the control law described above which ensures that, locally, the system converges to an equilibrium of the form (37) with $\boldsymbol{P}_e = m_1\tilde{v}_1\boldsymbol{e_1}$; see (Woolsey, 2001).

5 Final Remarks

We have applied a three-step approach to stabilize an underwater vehicle with coincident centers of buoyancy and gravity using three internal rotors. The first step involved shaping the system's inertia through feedback; steady long-axis translation was stabilized by choosing control gains to make the control-modified inertia matrix negative definite. The second step made use of a Lyapunov function developed in the first step in order to provide asymptotic stability. In the third step, we included a general model for the viscous force and torque on an ellipsoidal underwater vehicle and showed that the effect is to enhance stability, making the desired motion globally attractive.

We have also considered a more general problem in which the vehicle's center of gravity is below the center of buoyancy. We presented a control law which asymptotically stabilizes steady long-axis translation in the horizontal plane for a vehicle which is not subject to drag. Viscous forces and moments are discussed in (Woolsey, 2001).

As noted in Remark 9, the control law presented in Section 4 leads to an almost-Poisson, closed-loop system with a modified kinetic energy, a modified potential energy, and a modified dynamic structure. Also see (Woolsey & Leonard, 2000). The idea of modifying structure through feedback generalizes the idea of energy modification and leads to broader conditions for closed-loop stability. In the Hamiltonian setting, the IDA-PBC procedure allows for feedback modification of a system's dynamic structure, in addition to kinetic and potential shaping (**?**Ortega et al., 2002). Equivalently, this is allowed in the Lagrangian setting (Chang et al., 2002).

Besides providing broader conditions for closed-loop stability, modifying dynamic structure through feedback may alter a system's closed-loop dynamics in physically appealing ways. For example, Bloch et al. (1992) used feedback to modify the dynamics of a spacecraft with internal rotors to resemble the dynamics of a heavy top. Constructive methods for modifying the dynamic structure of a mechanical system through feedback are a topic of continuing research.

Acknowledgements. The authors thank H. Curtiss, P. Holmes, J. Kasdin, F. Lutze, and J. Marsden for informative discussions regarding this work. We also thank the anonymous reviewers for suggesting valuable improvements to the paper.

References

[1] Astolfi, A., Chhabra, D., and Ortega, R. (2001). Asymptotic stabilization of selected equilibria of the underactuated Kirchhoff's equations. *Proc. American Control Conf.*, pp. 4862–4867, Arlington, VA.

[2] Blankenstein, G., Ortega, R. and van der Schaft, A. (2002). The matching conditions of controlled Lagrangians and IDA-passivity based control. *Int. J. Control*, **75**(9), 645–665.

[3] Bloch, A. M., Chang, D. E., Leonard, N. E., & Marsden, J. E. (2001a). Controlled Lagrangians and the stabilization of mechanical systems II: Potential shaping. *IEEE Trans. Automatic Control*, **46**(10), 1556–1571.

[4] Bloch, A. M., Krishnaprasad, P. S., Marsden, J. E., Sánchez de Alvarez, G. (1992). Stabilization of rigid body dynamics by internal and external torques. *Automatica*, **28**(4), 745–756.

[5] Bloch, A. M., Leonard, N. E., & Marsden, J. E. (2000). Controlled Lagrangians and the stabilization of mechanical systems I: The first matching theorem. *IEEE Trans. Automatic Control*, **45**(12), 2253–2270.

[6] Bloch, A. M., Leonard, N. E., & Marsden, J. E. (2001b). Controlled Lagrangians and the stabilization of Euler-Poincaré mechanical systems. *Int. Journal of Robust and Nonlinear Control*, **11**(3), 191–214.

[7] Brockett, R. W. (1976). Control theory and analytic mechanics. *Lie Groups: History, Frontiers, and Applications, Vo. VII*, C. Martin & R. Hermann, Ed., Math Sci Press, pp. 1–49.

[8] Bullo, F. (2000). Stabilization of relative equilibria for underactuated systems on Riemannian manifolds. *Automatica*, **36**(12), 1819–1834.

[9] Chang, D. E., Bloch, A. M., Leonard, N. E., Marsden, J. E., & Woolsey, C. A. (2002). The equivalence of controlled Lagrangian and controlled Hamiltonian systems for simple mechanical systems. *ESAIM: Control, Optimisation, and Calculus of Variations*, to appear.

[10] Curtin, T. B., Bellingham, J. G., Catipovic, J., Webb, D. (1993). Autonomous oceanographic sampling networks. *Oceanography*, **6**, 86–94.

[11] Etkin, B. (1972). *Dynamics of Atmospheric Flight*, New York, NY: John Wiley and Sons.

[12] Fossen, T. (1995). *Guidance and Control of Ocean Vehicles*, New York, NY: John Wiley and Sons.

[13] Hoerner, S. F. (1965). *Fluid Dynamic Drag*, Midland Park, NJ: Published by the author.

[14] Hoerner, S. F. (1975). *Fluid Dynamic Lift*, H. V. Borst, Ed. Brick Town, NJ: Published by L. A. Hoerner.

[15] Holmes, P., Jenkins, J., & Leonard, N. E. (1998). Dynamics of the Kirchhoff equations I: Coincident centers of gravity and buoyancy. *Physica D*, **118**, 311–342.

[16] Jalnapurkar, S. M. & Marsden, J. E. (2000). Stabilization of relative equilibria. *IEEE Trans. Automatic Control*, **45**(8), 1483–1491.

[17] Krishnaprasad, P. S. (1985). Lie-Poisson structures, dual-spin spacecraft and asymptotic stability. *Nonlinear Analysis: Theory, Methods, and Applications*, **9**(10), 1011–1035.

[18] Lamb, H. (1932). *Hydrodynamics*, New York, NY: Dover.

[19] Leonard, N. E. (1997a). Stability of a bottom-heavy underwater vehicle. *Automatica*, **33**(3), 331–346.

[20] Leonard, N. E. (1997b). Stabilization of underwater vehicle dynamics with symmetry-breaking potentials. *Systems and Control Letters*, **32**, 35–42.

[21] Leonard, N. E. & Graver, J. G. (2001). Model-based feedback control of autonomous underwater gliders. *IEEE Journal of Oceanic Engineering*, **26**(4), 633–645.

[22] Leonard, N. E. & Woolsey, C. A. (1998). Internal actuation for intelligent underwater vehicle control. *Proc. Tenth Yale Workshop on Adaptive and Learning Systems*, pp. 295–300, New Haven, CT.

[23] Marsden, J. E. & Ratiu, T. S. (1994). *Introduction to Mechanics and Symmetry*, New York: Springer-Verlag.

[24] Ortega, R. Spong, M. W., Gómez-Estern, F., & Blankenstein, G. (2002). Stabilization of underactuated mechanical systems via interconnection and damping assignment. *IEEE Trans. Automatic Control*, **?**(?), ?–?.

[25] Cannas da Silva, A. & Weinstein, A. (1999). *Geometric Models for Noncommutative Algebras*, American Mathematical Society.

[26] van der Schaft, A. J. (1986). Stabilization of Hamiltonian systems. *Nonlinear Analysis: Theory, Methods, and Applications*, **10**(10), 1021–1035.

[27] Wetzel, T. G., & Simpson, R. L. (1998). Unsteady crossflow separation location measurements on a maneuvering 6:1 prolate spheroid. *AIAA Journal*, **36**(11), 2063–2071.

[28] Woolsey, C. A. (2001). *Energy Shaping and Dissipation: Underwater Vehicle Stabilization Using Internal Rotors*, Ph.D. Thesis, Princeton University. Available at: www.aoe.vt.edu/~cwoolsey/papers/thesis.html

[29] Woolsey, C. A. & Leonard, N. E. (1999a). Underwater vehicle stabilization by internal rotors. *Proc. American Control Conf.*, pp. 3417–3421, San Diego, CA.

[30] Woolsey, C. A. & Leonard, N. E. (1999b). Global asymptotic stabilization of an underwater vehicle using internal rotors. *Proc. IEEE Conf. Decision and Control*, pp. 2527–2532, Phoenix, AZ.

[31] Woolsey, C. A. & Leonard, N. E. (2000). Modification of Hamiltonian structure to stabilize an underwater vehicle. *Proc. IFAC Workshop on Lagrangian and Hamiltonian Methods for Nonlinear Control*, N. E. Leonard and R. Ortega, Ed., Pergamon, pp. 175-176.