# Underwater Vehicle Stabilization by Internal Rotors [1]

Craig A. Woolsey and Naomi Ehrich Leonard
Department of Mechanical & Aerospace Engineering
Princeton University
Princeton, NJ 08544
{cwoolsey,naomi}@princeton.edu

**Abstract**

Internal rotors may usefully complement more conventional means of underwater vehicle actuation by extending the operating regime and improving reliability. In this paper we describe stabilizing control laws for an underwater vehicle using internal rotors. Our stabilizing control laws consist of two terms: the first term addresses the conservative part of the system by shaping energy to yield Lyapunov stability, while the second term adds dissipation to ensure asymptotic stability to the motion of interest.

## 1 Introduction

Underwater vehicles are typically equipped with propellers and fins for motion control. However, the growing demand for autonomous underwater vehicles (AUVs) and the expanding range of conditions under which they must perform creates interest in exploring complementary and/or alternative means of actuation.

We consider here the use of internal rotors as a promising complement. For example, a long, slender vehicle moving along its long axis is typically stabilized using fins. At low velocities, however, fins lose their control authority and can no longer stabilize the vehicle. Internal rotors, which provide control by momentum exchange, could be used for stabilization in the low velocity regime, thereby extending the vehicle's range of operating conditions. Internal rotors are also protected from the corrosive seawater environment and they do not create drag.

We use a two-step approach to design a control law that stabilizes the dynamics of an underwater vehicle using internal rotors. In the first step, we treat the vehicle dynamics as a Hamiltonian system, and we design a control law that stabilizes a desired steady motion while preserving the Hamiltonian structure. The control shapes the kinetic energy (Hamiltonian) of the closed-loop system; see [1, 2, 3] for further background. In the second step, we add dissipative feedback to ensure asymptotic stability to the motion of interest. Making use of the global phase space structure studied in [4], we pick control gains that yield a large region of attraction. In future work we will also consider dissipation associated with viscous effects of the fluid.

In Section 2 we describe the dynamics of an underwater vehicle, and in Section 3 we revise our model to include internal rotors. In Section 4 a feedback law is chosen which stabilizes long axis translation. The equilibrium is made asymptotically stable using an additional feedback term derived in Section 5. We conclude in Section 6.

## 2 Underwater Vehicle Dynamics

We consider a vehicle model based on Kirchhoff's equations which describe a six degree-of-freedom, neutrally buoyant rigid body in an infinite volume of irrotational, incompressible, inviscid fluid that is at rest at infinity [7]. Forces such as viscous effects and external control inputs can be added to this basic model as desired. For ease of presentation, we assume the vehicle is an ellipsoid with uniformly distributed mass.

Let $\Omega$ and $v$ be the vehicle angular and linear velocity vectors in body coordinates, where the ellipsoid principal axes serve as the body coordinate axes. Then, the angular momentum $\Pi$ and linear momentum $P$ vectors are given by

$$\Pi = I\Omega, \quad P = Mv.$$

The diagonal matrix $I$ is the sum of the body inertia and the *added inertia* from the potential flow model of the fluid. Similarly, the diagonal matrix $M$ is the sum of the mass of the body $m$ multiplied by the identity matrix and the *added mass matrix*.

In the absence of external forces and torques, the dynamic equations of motion (Kirchhoff's equations) are

$$\begin{aligned} \dot{\Pi} &= \Pi \times \Omega + P \times v \\ \dot{P} &= P \times \Omega. \end{aligned} \tag{2.1}$$

Equations (2.1) describe Lie-Poisson (non-canonical

[1]Research partially supported by the National Science Foundation under grant BES-9502477 and by the Office of Naval Research under grants N00014-96-1-0052 and N00014-98-1-0649.

Hamiltonian) dynamics [10]; they are equivalent to $\dot{\nu} = \{\nu, H\}$, where $\nu$ is a component of $\Pi$ or $P$ and the Poisson bracket is defined as

$$\{G, K\}(\Pi, P) = \nabla G \cdot \begin{pmatrix} \hat{\Pi} & \hat{P} \\ \hat{P} & 0 \end{pmatrix} \nabla K$$

for differentiable functions $G$ and $K$. The Hamiltonian $H$ is the total kinetic energy of the body-fluid system

$$H(\Pi, P) = \frac{1}{2}\Pi \cdot I^{-1}\Pi + \frac{1}{2}P \cdot M^{-1}P. \tag{2.2}$$

The two Casimir functions $C_1 = \Pi \cdot P$ and $C_2 = \frac{1}{2}\|P\|^2$ commute under the Poisson bracket and are thus conserved along the equations of motion.

Let $L_i$ be the length of the $i$th principal axis of the ellipsoid. We will assume that $L_1 > L_2 > L_3$. Let the diagonal elements of $M$ and $I$ be $(m_1, m_2, m_3)$ and $(I_1, I_2, I_3)$, respectively. For the given ordering of axis lengths, it is always true that $m_3 > m_2 > m_1$. However, the inertia elements may be ordered $I_3 > I_2 > I_1$ or $I_2 > I_3 > I_1$ or $I_2 > I_1 > I_3$, depending on the relative lengths of the semiaxes [8, 4].

The model (2.1) has several families of (relative) equilibria [4]. There are three two-parameter families of equilibria corresponding to steady translation along and rotation about a vehicle principal axis. We refer to these as pure-mode equilibria. It was shown in [4] that there may also exist non-principal axis steady motions referred to as mixed-mode equilibria. The existence of these equilibria depends on the inertia ordering as well as the ratio $C_1/C_2$.

Stability of the equilibria of (2.1) can be studied using the energy-Casimir method, applicable to Lie-Poisson systems. The method consists of defining a (Lyapunov) function $H_{\Phi,\phi} = H + \Phi(C_i) + \phi_j(c_j)$, where $H$ is the Hamiltonian, $C_i$ are Casimirs, $c_j$ are other constants of motion. Showing that smooth functions $\Phi(\cdot)$ and $\phi_j(\cdot)$ can be found so that the equilibrium is a maximum or minimum of $H_{\Phi,\phi}$ proves nonlinear stability [9].

Since the 1 axis is the long axis, the pure 1 mode equilibrium is a practical (streamlined) vehicle motion. Applying the energy-Casimir method to this equilibrium fails to provide conditions for stability. In fact, it is well known that translation of an ellipsoid along its long axis through a fluid is unstable [7].

## 3 Dynamics of a Vehicle with Internal Rotors

We model an ellipsoidal vehicle with three internal rotors. The model is analogous to the models of spacecraft with internal rotors presented in [6] and [1]. We assume that each of the three rotors is symmetric and spins about its axis of symmetry under the influence of a control torque. The rotors are mounted orthogonally within the vehicle so that each rotor's spin axis is collinear with a vehicle principal axis and so that the CG of the vehicle is still at the CB; see Figure 3.1.

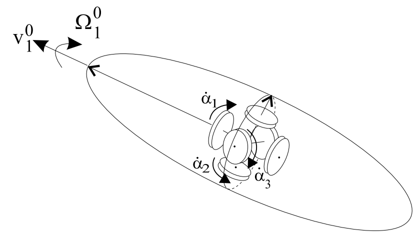

**Figure 3.1:** Vehicle with three internal rotors, each comprised of two rigidly coupled disks.

Let the diagonal matrix with diagonal elements $(J_1^i, J_2^i, J_3^i)$ denote the inertia of the rotor which spins about the $i$th vehicle axis ($i = 1, 2,$ or $3$). Assume that this rotor spins at a rate $\dot{\alpha}_i$ relative to the vehicle as depicted in Figure 3.1. Define

$$\lambda_j = I_j + J_j^1 + J_j^2 + J_j^3, \quad j = 1, 2, 3.$$

Then the locked inertia matrix is $\Lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$. It is also convenient to define the matrix of rotor spin axis inertias $J_r = \text{diag}(J_1^1, J_2^2, J_3^3)$ and the matrix $\bar{I} = \text{diag}(\bar{I}_1, \bar{I}_2, \bar{I}_3) = \Lambda - J_r$.

Define $\Omega_r = (\dot{\alpha}_1, \dot{\alpha}_2, \dot{\alpha}_3)^T$. Then

$$\Pi = \Lambda\Omega + J_r\Omega_r, \quad P = Mv, \quad l = J_r(\Omega + \Omega_r).$$

$\Pi$ is the total (body plus rotors plus fluid) angular momentum, $P$ is the total linear momentum, and $l = (l_1, l_2, l_3)^T$ where $l_i$ is the total momentum of the $i$th rotor about its spin axis. The equations of motion are

$$\begin{aligned} \dot{\Pi} &= \Pi \times \Omega + P \times v \\ \dot{P} &= P \times \Omega \\ \dot{l} &= u. \end{aligned} \tag{3.1}$$

The control is $u = (u_1, u_2, u_3)^T$ where $u_i$ is the torque applied to the $i$th internal rotor about its spin axis. The quantities $\Pi \cdot P$ and $\frac{1}{2}\|P\|^2$ are conserved for any choice of $u$; internal rotors cannot affect the inertial angular or linear momentum of the system.

## 4 Hamiltonian Stabilization

In this section, we apply feedback which stabilizes a closed-loop relative equilibrium corresponding to pure translation along the vehicle long axis. We choose a control law such that the closed-loop system remains Hamiltonian. The closed-loop Hamiltonian is a control-dependent modification of the open-loop Hamiltonian.

The energy-Casimir method then provides a control-dependent Lyapunov function which can be used to find conditions on the control to ensure closed-loop stability.

We choose a control of the form

$$u = K\dot{\Pi} = K\big((\Lambda\Omega + J_r\Omega_r) \times \Omega + Mv \times v\big) \quad (4.1)$$

where $K$ is a matrix of control gains. For simplicity, we choose $K = \mathrm{diag}(k_1, k_2, k_3)$. The control law defined here is inspired by the satellite stabilization problem in [1] and can be determined by applying the algorithmic *method of controlled Lagrangians* described in [2, 3].

Define the change of variables

$$\zeta = (\mathcal{I} - K)^{-1}(l - K\Pi) \quad (4.2)$$

where $\mathcal{I}$ is the 3 x 3 identity matrix and $k_i \neq 1$ for $i = 1, 2$, and 3. Then, $\Pi - \zeta = Z\Omega$ where

$$Z = (\mathcal{I} - K)^{-1}\bar{I}. \quad (4.3)$$

The closed-loop equations of motion are

$$\begin{aligned} \dot{\Pi} &= \Pi \times \Omega + P \times v \\ \dot{P} &= P \times \Omega \\ \dot{\zeta} &= 0. \end{aligned} \quad (4.4)$$

As intended, the equations (4.4) describe Lie-Poisson dynamics:

$$\begin{pmatrix} \dot{\Pi} \\ \dot{P} \\ \dot{\zeta} \end{pmatrix} = \begin{pmatrix} \hat{\Pi} & \hat{P} & 0 \\ \hat{P} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \nabla H_C \quad (4.5)$$

where the new Hamiltonian $H_C$ depends on the control gains and is a modification of the Hamiltonian (2.2) for the uncontrolled system:

$$H_C(\Pi, P, \zeta) = \frac{1}{2}(\Pi - \zeta) \cdot Z^{-1}(\Pi - \zeta) + \frac{1}{2} P \cdot M^{-1} P. \quad (4.6)$$

$Z$ is a diagonal matrix whose diagonal elements $Z_1, Z_2$, and $Z_3$ depend on the control gains by (4.3). There are five independent Casimirs for this system; $C_1 = \Pi \cdot P$, $C_2 = \frac{1}{2}\|P\|^2$ and each component of $\zeta$ are conserved.

Pure-mode equilibria of the uncontrolled system are also equilibria of the closed-loop system with $\zeta_e \parallel \Pi_e \parallel P_e$. We focus on the two-parameter family of pure 1 mode equilibria

$$\Pi_e = \zeta_e = \Pi_1^0 \, e_1, \quad P_e = P_1^0 \, e_1 \quad (4.7)$$

with $e_1$ the unit vector along the body's 1 axis. The equilibrium (4.7) corresponds to vehicle translation along the long axis with the 1 axis rotor spinning at a constant rate and the body not spinning ($\Omega_e = 0$).

**Theorem 4.1** *Consider the vehicle with three internal rotors described by (3.1) and the control $u$ given by (4.1). Choose $k_i > 1$ so that $Z_i < 0$ for $i = 1, 2$, and 3. Then the equilibrium (4.7) is Lyapunov stable.*

*Proof:* The theorem follows by applying the energy-Casimir method to the conserved quantity

$$H_\Phi = H_C(\Pi, P, \zeta) + \Phi(C_1, C_2, \zeta_1, \zeta_2, \zeta_3). \quad (4.8)$$

The method provides $\Phi$ and Lyapunov function

$$\begin{aligned} H_\Phi = {} & H_C(\Pi, P, \zeta) - \tfrac{1}{m_1} C_2 + \tfrac{1}{2} a_1 (C_1 - \Pi_1^0 P_1^0)^2 \\ & + a_2 (C_1 - \Pi_1^0 P_1^0) \cdot (C_2 - \tfrac{1}{2}(P_1^0)^2) \\ & + \tfrac{1}{2} a_3 (C_2 - \tfrac{1}{2}(P_1^0)^2)^2 + \tfrac{1}{2} a_4 \zeta_2^2 + \tfrac{1}{2} a_5 \zeta_3^2 \end{aligned} \quad (4.9)$$

with scalar constants

$$a_1 = -\frac{1}{2}\frac{1}{Z_1 (P_1^0)^2}, \quad a_2 = -a_1 \left(\frac{\Pi_1^0}{P_1^0}\right),$$

$$a_3 = -3a_1 \frac{1 + \left(\Pi_1^0\right)^2}{\left(P_1^0\right)^2}, \quad a_4 < 0, \quad a_5 < 0.$$

With $Z_i < 0$ for $i = 1, 2$, and 3, the equilibrium (4.7) is a maximum of $H_\Phi$, and stability follows by the energy-Casimir method.

**Remark 4.2** The pure 2 and pure 3 mode equilibria are unstable (saddle points) for this choice of $Z$. However, $\mathcal{F}_1 = \{(\Pi, P, \zeta) \mid \Pi_2 = \Pi_3 = P_1 = \zeta_2 = \zeta_3 = 0\}$ is invariant, and on $\mathcal{F}_1$ the pure 2 mode is stable. This observation plays a role in Section 5 where we add feedback dissipation to achieve asymptotic stability and we seek a large region of attraction.

## 5 Feedback-controlled Dissipation

In this section, we introduce feedback-controlled dissipation in order to asymptotically stabilize the equilibrium. We begin by replacing $u$ in the original equations of motion (3.1) with $u = u_s + u_d$ where $u_s$ given by (4.1) provides Lyapunov stability as in Section 4 and $u_d$ will be chosen to add dissipation. Under the change of variables (4.2), the equations of motion become

$$\begin{aligned} \dot{\Pi} &= \Pi \times \Omega + P \times v \\ \dot{P} &= P \times \Omega \\ \dot{\zeta} &= (\mathcal{I} - K)^{-1} u_d \end{aligned} \quad (5.1)$$

where, as in Section 4, $\Omega = Z^{-1}(\Pi - \zeta)$.

With $u_d = 0$, the equations (5.1) are equivalent to the conservative equations (4.4). We choose $H_\Phi$ defined by (4.9) as a natural Lyapunov function candidate for the dissipative equations (5.1). Since the equilibrium (4.7) is a local maximum of $H_\Phi$, proving asymptotic stability of the dissipative equations involves finding a control $u_d$ for which $\dot{H}_\Phi \geq 0$. We find that

$$\dot{H}_\Phi = \left( -Z^{-1}(\Pi - \zeta) + \begin{pmatrix} 0 \\ a_4 \zeta_2 \\ a_5 \zeta_3 \end{pmatrix} \right) \cdot (\mathcal{I} - K)^{-1} u_d \quad (5.2)$$

where $a_4$ and $a_5$ are arbitrary negative constants as defined in the proof of Theorem 4.1. Taking

$$u_d = (\mathcal{I} - K)K_d \left( -Z^{-1}(\Pi - \zeta) + \begin{pmatrix} 0 \\ a_4\zeta_2 \\ a_5\zeta_3 \end{pmatrix} \right), \tag{5.3}$$

with $K_d$ a positive definite $3 \times 3$ dissipative control gain matrix, makes $\dot{H}_\Phi \geq 0$.

Let $\omega$ be a compact subset of $\mathcal{D} = \{(\Pi, P, \zeta) \mid \Pi \cdot P = C_1\ ,\ \frac{1}{2}\|P\|^2 = C_2\}$ that is positively invariant under the dynamics (5.1). By LaSalle's theorem (see [5], for example), solutions starting in $\omega$ go to the largest invariant set $\mathcal{M}$ in $E = \{(\Pi, P, \zeta) \in \omega \mid \dot{H}_\Phi = 0\}$. It can be shown that $\mathcal{M}$ consists only of closed-loop equilibria regardless of the choice of $\omega$.

Equations (5.1) with (5.3) admit all three pure mode equilibria. There are also mixed 2-3 mode equilibria [4] if

$$\frac{\left(\frac{1}{m_2} - \frac{1}{m_3}\right)}{(Z_2 + \frac{1}{a_4}) - (Z_3 + \frac{1}{a_5})} > 0.$$

The terms $Z_2, Z_3, a_4$, and $a_5$ are all control parameters which must be negative. Choosing these parameters such that

$$Z_2 + \frac{1}{a_4} < Z_3 + \frac{1}{a_5}$$

ensures that only pure-mode equilibria are possible.

**Remark 5.1** If $\Omega \neq 0$, then the pure 2 mode equilibria are a two-parameter family of the form

$$\Pi_e = (1 + a_4 Z_2)\zeta_e = \Pi_2^0\ e_2, \quad P_e = P_2^0\ e_2.$$

If $\Omega = 0$, we have $\Pi_e = \zeta_e = \Pi_1^0\ e_1$ and $P_e = P_2^0\ e_2$. The pure 3 mode equilibria are defined analogously.

To conclude asymptotic stability of the pure 1 mode equilibrium, we must choose $\omega$ such that the other pure mode equilibria do not live in $\mathcal{M}$. For pure 1 axis translation, $H_\Phi = 0$, its maximum value. Evaluated at a pure 2 or 3 mode equilibrium, $H_\Phi \leq -(\frac{1}{m_1} - \frac{1}{m_2})C_2 < 0$. Define a constant $c = -(1-\epsilon)(\frac{1}{m_1} - \frac{1}{m_2})C_2$ where $0 < \epsilon \ll 1$ and let

$$\omega = \{(\Pi, P, \zeta) \in \mathcal{D} \mid H_\Phi \geq c\}. \tag{5.4}$$

Then the equilibria (4.7) with $P_1^0 = \pm\sqrt{2C_2}$ and $\Pi_1^0 = C_1/P_1^0$ are the only (two) equilibria in $\omega$. The set $\omega$ is compact because $H_\Phi$ is radially unbounded. The set is positively invariant because $H_\Phi$ is nondecreasing. Referring to the definition (4.9) of $H_\Phi$, we note that by choosing smaller magnitudes of $Z_i$ ($i = 1, 2$, and 3), $a_4$, and $a_5$, we may increase the region $\omega$.

We note that $\omega$ excludes states for which $P_1 = 0$.

**Lemma 5.2** $\{(\Pi, P, \zeta) \in \mathcal{D} \mid P_1 = 0\} \cap \omega = \emptyset$.

*Proof:* ¿From (4.9), one can calculate that if $P_1 = 0$ then $H_\Phi \leq -(\frac{1}{m_1} - \frac{1}{m_2})C_2$. This implies that $\omega$ given by (5.4) excludes states for which $P_1 = 0$.

**Theorem 5.3** *Any solution to the equations (5.1) which starts in $\omega$ at time $t = 0$ with $u_d$ given by (5.3) and with $P_1(0) > 0$ goes to*

$$\Pi_e = \zeta_e = \frac{C_1}{\sqrt{2C_2}}\ e_1, \quad P_e = \sqrt{2C_2}\ e_1$$

*as $t \to \infty$. If $P_1(0) < 0$ the solution goes to $\{-\Pi_e, -P_e, -\zeta_e\}$ as $t \to \infty$.*

*Proof:* The fact that the state goes to a pure 1 mode equilibrium follows directly from Lasalle's theorem. Now assume that $P_1(0) > 0$ and $P_1 \to -\sqrt{2C_2} < 0$. By continuity of the solution, $P_1(\bar{t}) = 0$ at some time $t = \bar{t} > 0$. This is a contradiction by Lemma 5.2 because the state begins in $\omega$ and $\omega$ is positively invariant. The case of $P_1(0) < 0$ follows similarly.

Theorem 5.3 indicates that the vehicle approaches a pure 1 mode equilibrium. The final magnitudes of $v_1$ and $\dot{\alpha}_1$ are determined by the conservation laws.

Numerical simulations were performed for a vehicle modeled as a neutrally buoyant ellipsoid with axis lengths $L_1 = 0.4572$ m, $L_2 = 0.3048$ m, and $L_3 = 0.1524$ m. For the density of water $\rho = 1000$ kg/m$^3$, $m_1 = 13.2$ kg, $m_2 = 15.2$ kg and $m_3 = 25.6$ kg. The vehicle body has uniformly distributed mass and each internal rotor is modeled as a pair of rigidly coupled thin disks each of mass $m_{disk} = 0.25$ kg and radius $r = 0.0254$ m spinning about a given principal axis. Each disk is located a distance $d = .0381$ m along the principal axis from the vehicle CB in either direction (see Figure 3.1).

It is desired to stabilize the vehicle with no body angular velocity, $\Omega_d = 0$ rad/s, and a translational velocity $v_d = 0.1\ e_1$ m/s. The control gains are chosen to satisfy the requirements developed in the preceding analysis. Specifically, $K = \text{diag}(4.3, 8.5, 10)$ so that $Z = \text{diag}(-0.0176, -0.0171, -0.0169)$ kg-m$^2$. Also, $K_d = \text{diag}(0.1, 0.1, 0.05)$, $a_4 = -2$, and $a_5 = -5$.

Figure 5.1 shows the velocity response to an initial perturbation from the desired equilibrium:

$$\begin{aligned} \Omega(0) &= (0.01, 0.01, 0.01)^T \quad \text{rad/s}, \\ v(0) &= (0.09, 0.02, 0.02)^T \quad \text{m/s and}, \\ \Omega_r(0) &= (1, 1, 1)^T \quad \text{rad/s}. \end{aligned}$$

For the given initial velocities, $H_\Phi = -0.0053$ J initially. Since $\omega = \{(\Pi, P, \zeta) \in \mathcal{D} \mid H_\Phi \geq c\}$ where

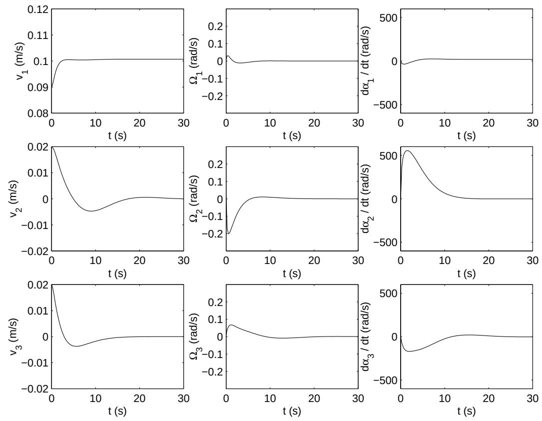

**Figure 5.1:** Closed-loop response to a perturbation.

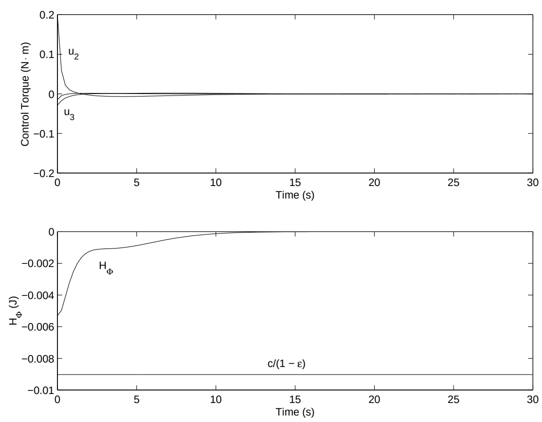


**Figure 5.2:** Control torques and the Lyapunov function.

$\frac{1}{1-\epsilon}c = -(\frac{1}{m_1} - \frac{1}{m_2})C_2 = -0.0090$ J, the initial state is in $\omega$ for $\epsilon$ small enough; see Figure 5.2.

Figure 5.1 shows $v$ approaching the desired velocity. Because $P_1(0) > 0$, $v_1 \to \sqrt{2C_2}/m_1 = 0.1$ m/s. The final magnitude of $v_1$ is as desired only because initial conditions were chosen such that $C_2 = \frac{1}{2}\|P_d\|^2$. On the other hand, $C_1 \neq \Pi_d \cdot P_d$. Accordingly, all components of the body and rotor angular velocity approach zero except for $\dot{\alpha}_1$ which takes the appropriate magnitude to maintain $C_1$ at its initial value.

## 6 Conclusions

We have shown how to asymptotically stabilize steady, long-axis translational motion of an underwater vehicle using internal rotors. The control law consists of a kinetic energy shaping term plus a dissipative feedback term. Some investigation of global phase space structure led to conditions on control gains to ensure a reasonably sized region of attraction for the steady motion of interest. For example, control gains were selected to prevent the existence of mixed-mode equilibria which in the uncontrolled system are known to lead to vehicle turning motions [4]. Further work on using the control law to manipulate the global phase space structures to advantage is ongoing.

Although physical dissipation was not considered in this paper, one might reasonably expect that damping would be destabilizing because the equilibrium is a relative maximum of the Hamiltonian. Preliminary analysis verifies this intuition but also indicates that suitable choices of the control parameters can compensate for physical damping. The problem of physical dissipation is a focus of our continuing work.

## References

[1] A. M. Bloch, P. S. Krishnaprasad, J. E. Marsden, and G. Sánchez de Alvarez. Stabilization of rigid body dynamics by internal and external torques. *Automatica*, 28(4):745–756, 1992.

[2] A. M. Bloch, N. E. Leonard, and J. E. Marsden. Stabilization of mechanical systems using controlled Lagrangians. In *Proc. IEEE C.D.C.*, pages 2356–2361, San Diego, CA, December 1997.

[3] A. M. Bloch, N. E. Leonard, and J. E. Marsden. Matching and stabilization by the method of controlled Lagrangians. In *Proc. IEEE C.D.C.*, pages 1446–1451, Tampa, FL, December 1998.

[4] P. Holmes, J. Jenkins, and N. E. Leonard. Dynamics of the Kirchhoff equations I: Coincident centers of gravity and buoyancy. *Physica D*, 118:311–342, 1998.

[5] H. Khalil. *Nonlinear Systems.* Prentice Hall, Upper Saddle River, New Jersey, 2nd edition, 1996.

[6] P. S. Krishnaprasad. Lie-Poisson structures, dual-spin spacecraft and asymptotic stability. *Nonlinear Anal., Theory, Meth. & App.*, 9(10):1011–1035, 1985.

[7] H. Lamb. *Hydrodynamics.* Dover, New York, 6th edition, 1932.

[8] N. E. Leonard. Stability of a bottom-heavy underwater vehicle. *Automatica*, 33(3):331–346, 1997.

[9] J. E. Marsden and T. S. Ratiu. *Introduction to Mechanics and Symmetry.* Springer-Verlag, 1994.

[10] S.P. Novikov and I. Shmel'tser. Periodic solutions of Kirchhoff's equations for the free motion of a rigid body in a fluid and the extended theory of LSM. I. *Funct. Anal. Appl.*, 15:197–207, 1982.