Moving Mass Control for Underwater Vehicles

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Abstract

We present two reduced-dimensional, noncanonical Hamiltonian models for a neutrally buoyant underwater vehicle coupled to an internal moving mass. It is expected that these models will be useful in designing nonlinear control laws for underwater gliders as well as for spacecraft, atmospheric re-entry vehicles, and other vehicles which use internal moving mass actuators. To illustrate, we investigate stability of a steady underwater vehicle motion using potential shaping feedback with a moving mass actuator.

1 Introduction

Internal degrees of freedom can significantly influence a vehicle's dynamics, either adversely or advantageously. Simple models for the coupled dynamics between a vehicle and internal moving parts can provide a great deal of insight into stability and control design. For example, Abzug modeled fuel sloshing in an aircraft as an oscillating point mass and showed that this simple oscillator can excite undesired lateral-directional dynamics [1]. Damped point mass oscillators have also been used to study bifurcations in gyrostats [5].

Internal moving elements can also provide actuation. Internal actuators offer advantages over conventional actuators that make them appealing in many vehicle applications. In space applications, internal actuators are useful because they can be powered using readily generated electricity rather than costly propellant and they can be housed completely within the spacecraft, away from the extreme environment of space. Internal actuators are appealing for atmospheric re-entry vehicles because they are impervious to the high temperatures and forces of hypersonic re-entry [11, 12].

For underwater vehicles, internal actuators are appealing because they can be used at low velocity, where fins lose their control authority, and they are impervious to corrosion and biological fouling [14]. An intriguing application for moving mass control in underwater vehicles is the underwater glider [8]. An underwater glider is a winged underwater vehicle which propels itself by alternately adjusting its center of mass and the buoyant force acting on it. Underwater gliding promises to provide an efficient alternative to conventional underwater vehicle propulsion. It is expected that the vehicle models presented here will be useful for studying the nonlinear dynamics of underwater gliding and for designing control laws.

We develop two models for a neutrally buoyant vehicle with a moving mass from two different views of the system dynamics. For the first model, the dynamics of the rigid body and point mass are viewed independently; the equations of motion involve the translational and angular momenta of the rigid body and the momentum of the point mass. The force of interaction between these objects is treated as a control. This view makes the effect of internal actuation on the base body obvious. For the second model, the system dynamics are described in terms of the total system momen-

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tum and the point mass momentum. Because the control force is an internal force, total momentum is conserved for any choice of control. This fact is useful for studying stability of steady motions.

The two vehicle models can be described as underactuated, non-canonical Hamiltonian control systems. Hamiltonian control systems, and their Lagrangian equivalents, exhibit a great deal of structure which can be exploited for stability analysis and control design. For example, the method of controlled Lagrangians, introduced in [4], involves shaping the kinetic and potential energy of an underactuated mechanical system in order to stabilize a desired equilibrium. More generally, one may shape the dynamic structure defining Lagrange's or Hamilton's equations, in addition to the total energy. In the Hamiltonian setting, the idea of energy and structure shaping has been referred to as interconnection and damping assignment, passivity-based control (IDA-PBC) [10, 2]. In their most general forms, the method of controlled Lagrangians and IDA-PBC are equivalent [3]. These methods provide nonlinear control laws which respect the essential physics of mechanical control systems. They promise to provide controllers which are effective over larger regions of phase space and which may prove to have good robustness properties.

Energy-based control methods benefit from the existence of tools for constructive nonlinear stability analysis such as the energy-Casimir method and the energy-momentum method. (See [9], for example.) Lyapunov functions developed using these methods can be used to design asymptotically stabilizing control laws, to estimate regions of attraction for feedback-stabilized equilibria, and to study robustness. We illustrate the utility of the models described here by designing feedback to stabilize a steady underwater vehicle motion.

2 Kinematics

Figure 1 depicts a rigid body and a point mass immersed in an ideal fluid. An inertial coordinate frame $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ is fixed in space. A body coordinate frame $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ is fixed to a point in the body. These two coordinate frames are related by the proper rotation matrix $\boldsymbol{R}$ and the vector $\boldsymbol{b}$, expressed in inertial coordinates, which locates the body coordinate origin with respect to the inertial coordinate origin. A vector $\boldsymbol{x}_B$, expressed with respect to the body coordinate frame, can be converted to inertial coordinates by the relation

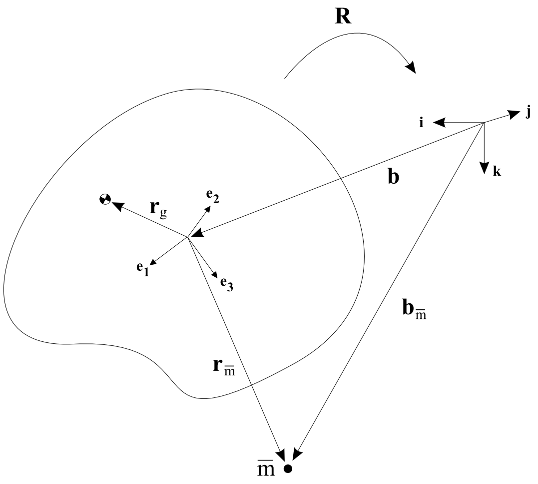

Figure 1: Rigid Body and Point Mass in a Fluid

$$\boldsymbol{x}_I = \boldsymbol{R}\boldsymbol{x}_B + \boldsymbol{b}.$$

The inertial vector $\boldsymbol{b}_{\bar{m}}$ locates the point mass $\bar{m}$ in inertial space. The system configuration is given by the three elements $(\boldsymbol{R}, \boldsymbol{b}, \boldsymbol{b}_{\bar{m}})$. (The configuration space is the product of Lie groups $SE(3) \times \mathbb{R}^3$.) Alternatively, one may describe the location of the point mass relative to the body coordinate origin. Let $\boldsymbol{r}_{\bar{m}} = \boldsymbol{R}^T(\boldsymbol{b}_{\bar{m}} - \boldsymbol{b})$ be the location of the point mass with respect to the body coordinate frame, written in body coordinates. The configuration of the system is equivalently given by the three elements $(\boldsymbol{R}, \boldsymbol{b}, \boldsymbol{r}_{\bar{m}})$.

Define the body angular velocity $\boldsymbol{\Omega}$ and the body translational velocity $\boldsymbol{v}$. Also, define the operator $\hat{\cdot}$ which converts a vector into a skew-symmetric matrix such that $\hat{\boldsymbol{x}}\boldsymbol{y} = \boldsymbol{x} \times \boldsymbol{y}$, where $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3$. The rigid body kinematics are given by

$$\begin{aligned} \dot{\boldsymbol{R}} &= \boldsymbol{R}\hat{\boldsymbol{\Omega}} \\ \dot{\boldsymbol{b}} &= \boldsymbol{R}\boldsymbol{v}. \end{aligned} \tag{1}$$

We also define the inertial velocity of the point mass $\bar{m}$ written in body coordinates

$$\boldsymbol{v}_{\bar{m}} = \boldsymbol{R}^T\dot{\boldsymbol{b}}_{\bar{m}}.$$

Differentiating the definition of $\boldsymbol{r}_{\bar{m}}$, we find

$$\begin{aligned}\dot{\boldsymbol{r}}_{\bar{m}} &= \dot{\boldsymbol{R}}^T(\boldsymbol{b}_{\bar{m}} - \boldsymbol{b}) + \boldsymbol{R}^T(\dot{\boldsymbol{b}}_{\bar{m}} - \dot{\boldsymbol{b}}) \\ &= -\hat{\boldsymbol{\Omega}}\boldsymbol{R}^T(\boldsymbol{b}_{\bar{m}} - \boldsymbol{b}) + \boldsymbol{R}^T(\boldsymbol{R}\boldsymbol{v}_{\bar{m}} - \boldsymbol{R}\boldsymbol{v}).\end{aligned}$$

Substituting $\boldsymbol{r}_{\bar{m}}$ and rearranging gives the familiar expression for velocity in a rotating coordinate frame,

$$\boldsymbol{v}_{\bar{m}} = \boldsymbol{v} + \hat{\boldsymbol{\Omega}}\boldsymbol{r}_{\bar{m}} + \dot{\boldsymbol{r}}_{\bar{m}}.$$

3 Dynamics: Case 1

Let $\tilde{\boldsymbol{\Pi}}$ represent the total angular momentum of the body/fluid system about the body coordinate origin, written in body coordinates. Similarly, let $\tilde{\boldsymbol{P}}$ represent the translational momentum of the body/fluid system in body coordinates. Finally, let $\tilde{\boldsymbol{P}}_{\bar{m}}$ represent the point mass momentum written in body coordinates. We may compute the momenta $\tilde{\boldsymbol{\Pi}}, \tilde{\boldsymbol{P}}$, and $\tilde{\boldsymbol{P}}_{\bar{m}}$ from the kinetic energy of the body/fluid/point mass system. Let $\boldsymbol{\eta}_1 = (\boldsymbol{\Omega}^T, \boldsymbol{v}^T, \boldsymbol{v}_{\bar{m}}^T)^T$. The kinetic energy of the rigid body and the point mass is

$$T_{\text{body/mass}} = \frac{1}{2}\boldsymbol{\eta}_1 \cdot \mathbb{M}_{\text{rb}_1}\boldsymbol{\eta}_1$$

where

$$\mathbb{M}_{\text{rb}_1} = \begin{pmatrix} \boldsymbol{J}_b & m_b\hat{\boldsymbol{r}}_g & \boldsymbol{0} \\ -m_b\hat{\boldsymbol{r}}_g & m_b\mathbb{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \bar{m}\mathbb{I} \end{pmatrix}.$$

$\boldsymbol{J}_b$ is the rigid body inertia, m_b is the rigid body mass, and $\bar{m}$ is the point mass. The vector $\boldsymbol{r}_g$ denotes the location of the body's center of gravity (CG) with respect to the body coordinate origin. (See Figure 1.) The kinetic energy of the fluid takes the form

$$T_{\text{fluid}} = \frac{1}{2}\boldsymbol{\eta}_1 \cdot \mathbb{M}_{\text{fluid}}\boldsymbol{\eta}_1$$

where

$$\mathbb{M}_{\text{fluid}} = \begin{pmatrix} \boldsymbol{\Theta}_{11}^f & \boldsymbol{\Theta}_{12}^f & \boldsymbol{0} \\ \boldsymbol{\Theta}_{21}^f & \boldsymbol{\Theta}_{22}^f & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{pmatrix}.$$

The submatrices $\boldsymbol{\Theta}_{ij}^f$ form the generalized *added inertia tensor* which accounts for the energy necessary to accelerate the fluid around the moving body [6]. Added inertia depends only on the body's external geometry and the density of the fluid. Of course, there is no added inertia or inertial coupling associated with the point mass. The total system kinetic energy is

$$T = T_{\text{body/mass}} + T_{\text{fluid}} = \frac{1}{2}\boldsymbol{\eta}_1 \cdot \mathbb{M}_1\boldsymbol{\eta}_1,$$

where $\mathbb{M}_1 = \mathbb{M}_{\text{rb}_1} + \mathbb{M}_{\text{fluid}} > 0$. The conjugate momentum to $\boldsymbol{\eta}_1$ is $\boldsymbol{\nu}_1 = \frac{\partial T}{\partial \boldsymbol{\eta}_1} = (\tilde{\boldsymbol{\Pi}}^T, \tilde{\boldsymbol{P}}^T, \tilde{\boldsymbol{P}}_{\bar{m}}^T)^T$ where

$$\begin{aligned}\tilde{\boldsymbol{\Pi}} &= (\boldsymbol{J}_b + \boldsymbol{\Theta}_{11}^f)\boldsymbol{\Omega} + \boldsymbol{\Theta}_{12}^f\boldsymbol{v}, \\ \tilde{\boldsymbol{P}} &= \boldsymbol{\Theta}_{21}^f\boldsymbol{\Omega} + (m_b\mathbb{I} + \boldsymbol{\Theta}_{22}^f)\boldsymbol{v}, \\ \tilde{\boldsymbol{P}}_{\bar{m}} &= \bar{m}\boldsymbol{v}_{\bar{m}}.\end{aligned}$$

We assume that the body-plus-point-mass system is neutrally buoyant,

$$m_b + \bar{m} = m,$$

where m is the mass of the fluid which is displaced by the body. Suppose that the only forces and torques which act are due to gravity and buoyancy and to an internal control force $\boldsymbol{u}_{\text{int}}$ applied to the point mass from the body. The equations of motion become

$$\begin{aligned}\dot{\boldsymbol{R}} &= \boldsymbol{R}\hat{\boldsymbol{\Omega}}, \\ \dot{\boldsymbol{b}} &= \boldsymbol{R}\boldsymbol{v}, \\ \dot{\boldsymbol{r}}_{\bar{m}} &= \boldsymbol{v}_{\bar{m}} - \boldsymbol{v} - \boldsymbol{\Omega} \times \boldsymbol{r}_{\bar{m}} \\ \dot{\tilde{\boldsymbol{\Pi}}} &= \tilde{\boldsymbol{\Pi}} \times \boldsymbol{\Omega} + \tilde{\boldsymbol{P}} \times \boldsymbol{v} + m_b g \boldsymbol{r}_g \times (\boldsymbol{R}^T\boldsymbol{k}) \\ &\quad -\boldsymbol{r}_m \times \boldsymbol{u}_{\text{int}}, \\ \dot{\tilde{\boldsymbol{P}}} &= \tilde{\boldsymbol{P}} \times \boldsymbol{\Omega} - \bar{m}g(\boldsymbol{R}^T\boldsymbol{k}) - \boldsymbol{u}_{\text{int}}, \\ \dot{\tilde{\boldsymbol{P}}}_{\bar{m}} &= \tilde{\boldsymbol{P}}_{\bar{m}} \times \boldsymbol{\Omega} + \bar{m}g(\boldsymbol{R}^T\boldsymbol{k}) + \boldsymbol{u}_{\text{int}}. \qquad (2)\end{aligned}$$

Notice (in the $\dot{\tilde{\boldsymbol{P}}}$ equation) that the body experiences a vertically upward force of magnitude $\bar{m}g$ while (in the $\dot{\tilde{\boldsymbol{P}}}_{\bar{m}}$ equation) the point mass experiences a vertically downward force of the same magnitude. We anticipate including internal control forces which couple the body and point mass as a neutrally buoyant, multi-body system.

The dynamic equations in (2) are invariant under translation of the inertial coordinate frame and rotation of the frame about vertical, assuming that the control force $\boldsymbol{u}_{\text{int}}$ is chosen to preserve this symmetry. One may write the dynamic equations as a forced, noncanonical Hamiltonian system. First, define

$$\boldsymbol{\Gamma} = \boldsymbol{R}^T \boldsymbol{k},$$

the unit vector pointing in the direction of gravity, expressed in body coordinates. Define the state vector $\boldsymbol{x}_1 = (\tilde{\boldsymbol{\Pi}}^T,\ \tilde{\boldsymbol{P}}^T,\ \boldsymbol{\Gamma}^T,\ \boldsymbol{r}_{\bar{m}}^T,\ \tilde{\boldsymbol{P}}_{\bar{m}}^T)^T$. The dynamic equations from (2) may be written

$$\dot{\boldsymbol{x}}_1 = \boldsymbol{\Lambda}_1(\boldsymbol{x}_1)\nabla H_1(\boldsymbol{x}_1) + \boldsymbol{G}_1(\boldsymbol{x}_1)\boldsymbol{u}_{\text{int}}, \qquad (3)$$

where the Hamiltonian is

$$H_1(\boldsymbol{x}_1) = \frac{1}{2}\boldsymbol{\nu}_1 \cdot \mathbb{M}_1^{-1}\boldsymbol{\nu}_1 - (m_b g \boldsymbol{r}_g + \bar{m} g \boldsymbol{r}_{\bar{m}}) \cdot \boldsymbol{\Gamma}, \quad (4)$$

the "Poisson tensor" is

$$\boldsymbol{\Lambda}_1 = \begin{pmatrix} \hat{\tilde{\boldsymbol{\Pi}}} & \hat{\tilde{\boldsymbol{P}}} & \hat{\boldsymbol{\Gamma}} & \hat{\boldsymbol{r}}_{\bar{m}} & \hat{\tilde{\boldsymbol{P}}}_{\bar{m}} \\ \hat{\tilde{\boldsymbol{P}}} & \mathbf{0} & \mathbf{0} & \mathbb{I} & \mathbf{0} \\ \hat{\boldsymbol{\Gamma}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hat{\boldsymbol{r}}_{\bar{m}} & -\mathbb{I} & \mathbf{0} & \mathbf{0} & \mathbb{I} \\ \hat{\tilde{\boldsymbol{P}}}_{\bar{m}} & \mathbf{0} & \mathbf{0} & -\mathbb{I} & \mathbf{0} \end{pmatrix}$$

and the input matrix is

$$\boldsymbol{G}_1(\boldsymbol{x}_1) = \begin{pmatrix} -\hat{\boldsymbol{r}}_{\bar{m}} \\ -\mathbb{I} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbb{I} \end{pmatrix}.$$

Remark. Choosing $\boldsymbol{u}_{\text{int}} = -\nabla V_{\text{int}}(\boldsymbol{r}_{\bar{m}})$ couples the point mass dynamics to those of the body while preserving the Hamiltonian nature of the system. The potential function V_{int} may simply be appended to the Hamiltonian given in (4). In this case, and the special case where $\boldsymbol{u}_{\text{int}} = \mathbf{0}$, the system (3) is Lie-Poisson. (See [9].)

Generically, the fifteen-dimensional Poisson tensor $\boldsymbol{\Lambda}_1$ has rank twelve. Its null space is spanned by the gradients of the three "Casimir functions"

$$\begin{aligned} \tilde{C}_1 &= \frac{1}{2}(\tilde{\boldsymbol{P}} + \tilde{\boldsymbol{P}}_{\bar{m}}) \cdot (\tilde{\boldsymbol{P}} + \tilde{\boldsymbol{P}}_{\bar{m}}) \\ \tilde{C}_2 &= \boldsymbol{\Gamma} \cdot (\tilde{\boldsymbol{P}} + \tilde{\boldsymbol{P}}_{\bar{m}}) \\ \tilde{C}_3 &= \frac{1}{2}\boldsymbol{\Gamma} \cdot \boldsymbol{\Gamma}. \end{aligned}$$

Casimirs are conserved along the flow of the Hamiltonian vector field, a fact which can be useful in studying stability of steady motions. In fact, one may easily verify that these functions are conserved regardless of the choice of $\boldsymbol{u}_{\text{int}}$.

4 Dynamics: Case 2

Rather than consider the momentum of the body/fluid system separately from that of the point mass, one may consider the total system momentum. Let $\boldsymbol{\Pi}$ represent the total angular momentum of the system about the body coordinate origin, written in body coordinates. Similarly, let $\boldsymbol{P}$ represent the total translational momentum in body coordinates. Finally, let $\boldsymbol{P}_{\bar{m}}$ represent the point mass momentum written in body coordinates. (Of course, $\boldsymbol{P}_{\bar{m}} = \tilde{\boldsymbol{P}}_{\bar{m}}$, but we re-define the point mass momentum to maintain consistent notation.) We may compute the body momenta $\boldsymbol{\Pi}, \boldsymbol{P}$, and $\boldsymbol{P}_{\bar{m}}$ from the system kinetic energy. Define $\boldsymbol{\eta}_2 = (\boldsymbol{\Omega}^T, \boldsymbol{v}^T, \dot{\boldsymbol{r}}_{\bar{m}}^T)^T$. Then the kinetic energy of the rigid body and the point mass may be rewritten as

$$T_{\text{body/mass}} = \frac{1}{2}\boldsymbol{\eta}_2 \cdot \mathbb{M}_{\text{rb}_2}\boldsymbol{\eta}_2, \qquad (5)$$

where

$$\mathbb{M}_{\text{rb}_2} = \begin{pmatrix} \boldsymbol{J}_b - \bar{m}\hat{\boldsymbol{r}}_{\bar{m}}\hat{\boldsymbol{r}}_{\bar{m}} & m_b\hat{\boldsymbol{r}}_g + \bar{m}\hat{\boldsymbol{r}}_{\bar{m}} & \bar{m}\hat{\boldsymbol{r}}_{\bar{m}} \\ -m_b\hat{\boldsymbol{r}}_g - \bar{m}\hat{\boldsymbol{r}}_{\bar{m}} & (m_b + \bar{m})\mathbb{I} & \bar{m}\mathbb{I} \\ -\bar{m}\hat{\boldsymbol{r}}_{\bar{m}} & \bar{m}\mathbb{I} & \bar{m}\mathbb{I} \end{pmatrix}.$$

The system kinetic energy is

$$T = T_{\text{body/mass}} + T_{\text{fluid}} = \frac{1}{2}\boldsymbol{\eta}_2 \cdot \mathbb{M}_2\boldsymbol{\eta}_2$$

where $\mathbb{M}_2 = \mathbb{M}_{\text{rb}_2} + \mathbb{M}_{\text{fluid}} > 0$. The conjugate momentum to $\boldsymbol{\eta}_2$ is $\boldsymbol{\nu}_2 = \frac{\partial T}{\partial \boldsymbol{\eta}_2} = (\boldsymbol{\Pi}^T, \boldsymbol{P}^T, \boldsymbol{P}_{\bar{m}}^T)^T$ where

$$\begin{aligned} \boldsymbol{\Pi} &= (\boldsymbol{J}_b + \boldsymbol{\Theta}_{11}^f)\boldsymbol{\Omega} + (m_b\hat{\boldsymbol{r}}_g + \boldsymbol{\Theta}_{12}^f)\boldsymbol{v} \\ &\quad + \boldsymbol{r}_{\bar{m}} \times \bar{m}(\boldsymbol{v} - \hat{\boldsymbol{r}}_{\bar{m}}\boldsymbol{\Omega} + \dot{\boldsymbol{r}}_{\bar{m}}), \\ \boldsymbol{P} &= (-m_b\hat{\boldsymbol{r}}_g + \boldsymbol{\Theta}_{21}^f)\boldsymbol{\Omega} + (m_b\mathbb{I} + \boldsymbol{\Theta}_{22}^f)\boldsymbol{v} \\ &\quad + \bar{m}(\boldsymbol{v} - \hat{\boldsymbol{r}}_{\bar{m}}\boldsymbol{\Omega} + \dot{\boldsymbol{r}}_{\bar{m}}), \\ \boldsymbol{P}_{\bar{m}} &= \bar{m}(\boldsymbol{v} - \hat{\boldsymbol{r}}_{\bar{m}}\boldsymbol{\Omega} + \dot{\boldsymbol{r}}_{\bar{m}}). \end{aligned}$$

Suppose that the only forces and torques which act are those due to gravity and buoyancy and to the internal control force $\boldsymbol{u}_{\text{int}}$. Since we have assumed $m = m_b + \bar{m}$, the external forces on the total system due to gravity and buoyancy are equal and opposite. The equations of motion are

$$\dot{\boldsymbol{R}} = \boldsymbol{R}\hat{\boldsymbol{\Omega}},$$

$$\begin{aligned}
\dot{\boldsymbol{b}} &= \boldsymbol{R}\boldsymbol{v}, \\
\dot{\boldsymbol{r}}_{\bar{m}} &= \frac{1}{\bar{m}}\boldsymbol{P}_{\bar{m}} - \boldsymbol{v} - \boldsymbol{\Omega} \times \boldsymbol{r}_{\bar{m}} \\
\dot{\boldsymbol{\Pi}} &= \boldsymbol{\Pi} \times \boldsymbol{\Omega} + \boldsymbol{P} \times \boldsymbol{v} + m_b g \boldsymbol{r}_g \times \boldsymbol{R}^T \boldsymbol{k} \\
&\quad + \bar{m} g \boldsymbol{r}_{\bar{m}} \times \boldsymbol{R}^T \boldsymbol{k} \\
\dot{\boldsymbol{P}} &= \boldsymbol{P} \times \boldsymbol{\Omega} \\
\dot{\boldsymbol{P}}_{\bar{m}} &= \boldsymbol{P}_{\bar{m}} \times \boldsymbol{\Omega} + \bar{m} g \boldsymbol{R}^T \boldsymbol{k} + \boldsymbol{u}_{\text{int}} \qquad (6)
\end{aligned}$$

These dynamics were specialized to the case of longitudinal motion of an underwater glider in [8].

Remark. That $\boldsymbol{u}_{\text{int}}$ does not enter the equations for $\dot{\boldsymbol{\Pi}}$ or $\dot{\boldsymbol{P}}$ reflects the fact that internal actuation cannot alter the total system momentum.

Because of symmetry, one may reduce the dynamic equations to a noncanonical Hamiltonian form. Let $\boldsymbol{x}_2 = (\boldsymbol{\Pi}^T,\ \boldsymbol{P}^T,\ \boldsymbol{\Gamma}^T, \boldsymbol{r}_{\bar{m}}^T,\ \boldsymbol{P}_{\bar{m}}^T)^T$. The dynamic equations given in (6) may be rewritten

$$\dot{\boldsymbol{x}}_2 = \boldsymbol{\Lambda}_2(\boldsymbol{x}_2)\nabla H_2(\boldsymbol{x}_2) + \boldsymbol{G}_2(\boldsymbol{x}_2)\, \boldsymbol{u}_{\text{int}}, \qquad (7)$$

where the Hamiltonian is

$$H_2(\boldsymbol{x}_2) = \frac{1}{2}\boldsymbol{\nu}_2 \cdot \mathbb{M}_2^{-1}\boldsymbol{\nu}_2 - (m_b g \boldsymbol{r}_g + \bar{m} g \boldsymbol{r}_{\bar{m}}) \cdot \boldsymbol{\Gamma}, \quad (8)$$

the fifteen-dimensional “Poisson tensor” is

$$\boldsymbol{\Lambda}_2(\boldsymbol{x}_2) = \begin{pmatrix} \hat{\boldsymbol{\Pi}} & \hat{\boldsymbol{P}} & \hat{\boldsymbol{\Gamma}} & \boldsymbol{0} & \boldsymbol{0} \\ \hat{\boldsymbol{P}} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \hat{\boldsymbol{\Gamma}} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \mathbb{I} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\mathbb{I} & \boldsymbol{0} \end{pmatrix},$$

and the input matrix is

$$\boldsymbol{G}_2(\boldsymbol{x}_2) = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \mathbb{I} \end{pmatrix}.$$

Generically, the fifteen-dimensional Poisson tensor $\boldsymbol{\Lambda}_2$ has rank twelve. Three Casimirs are

$$\begin{aligned}
C_1 &= \frac{1}{2}\boldsymbol{P} \cdot \boldsymbol{P} \\
C_2 &= \boldsymbol{\Gamma} \cdot \boldsymbol{P} \\
C_3 &= \frac{1}{2}\boldsymbol{\Gamma} \cdot \boldsymbol{\Gamma}.
\end{aligned}$$

Note that $C_1 = \tilde{C}_1,\ C_2 = \tilde{C}_2$, and $C_3 = \tilde{C}_3$.

5 Example

We anticipate that the two vehicle models presented here will be useful for investigating dynamics and control of underwater gliders. As a simple example, we stabilize steady long-axis descent of a prolate spheroid using potential-shaping feedback control.

For a prolate spheroid with uniformly distributed mass and with body coordinates fixed in the ellipsoid principal axes, the inertia matrix $\boldsymbol{J} = \boldsymbol{J}_b + \Theta_{11}^f$ and the mass matrix $\boldsymbol{M} = m\mathbb{I} + \Theta_{22}^f$ are diagonal and there is no coupling between angular and translational kinetic energy, i.e., $\mathbb{M}_1$ is diagonal. We assume without loss of generality that the symmetry axis is the 3-axis. In this case, $m_3 < m_2 = m_1$, where $\boldsymbol{M} = \text{diag}(m_1, m_2, m_3)$.

We apply the potential-shaping control law

$$\boldsymbol{u}_{\text{int}} = -\boldsymbol{K}\boldsymbol{r}_{\bar{m}},$$

where $\boldsymbol{K} = \text{diag}(k_1, k_2, k_3)$ is a control gain matrix parameterizing the artificial potential function

$$V(\boldsymbol{r}_{\bar{m}}) = \frac{1}{2}\boldsymbol{r}_{\bar{m}} \cdot \boldsymbol{K}\boldsymbol{r}_{\bar{m}}.$$

The closed-loop system is Hamiltonian with dynamics given by (3) and with the Hamiltonian

$$\begin{aligned}
H_1(\boldsymbol{x}_1) &= \frac{1}{2}\tilde{\boldsymbol{\Pi}} \cdot \boldsymbol{J}^{-1}\tilde{\boldsymbol{\Pi}} + \frac{1}{2}\tilde{\boldsymbol{P}} \cdot \boldsymbol{M}^{-1}\tilde{\boldsymbol{P}} \\
&\quad + \frac{1}{2\bar{m}}\tilde{\boldsymbol{P}}_{\bar{m}} \cdot \tilde{\boldsymbol{P}}_{\bar{m}} - \bar{m} g \boldsymbol{r}_{\bar{m}} \cdot \boldsymbol{\Gamma} + \frac{1}{2}\boldsymbol{r}_{\bar{m}} \cdot \boldsymbol{K}\boldsymbol{r}_{\bar{m}}.
\end{aligned}$$

Suppose that $k_3 = \frac{\bar{m}g}{\gamma}$. Then an equilibrium is

$$\begin{aligned}
&\tilde{\boldsymbol{\Pi}}_e = \boldsymbol{0}, \quad \tilde{\boldsymbol{P}}_e = m_3 v_3^0 \boldsymbol{e}_3, \quad \boldsymbol{\Gamma}_e = \boldsymbol{e}_3, \\
&\boldsymbol{r}_{\bar{m}_e} = \gamma \boldsymbol{e}_3, \quad \tilde{\boldsymbol{P}}_{\bar{m}_e} = \bar{m} v_3^0 \boldsymbol{e}_3 \qquad . \quad (9)
\end{aligned}$$

The equilibrium (9) corresponds to steady, long axis descent at speed v_3^0 with the point mass $\bar{m}$ a distance γ below the body coordinate origin.

Theorem 5.1 *The equilibrium (9) is stable if*

$$\begin{aligned}
\gamma &> \frac{(m_1 - m_3)(v_3^0)^2}{\bar{m}g}, \quad \text{and} \\
k_i &> \frac{(\bar{m}g)^2}{\bar{m}g\gamma - (m_i - m_3)(v_3^0)^2}, \quad i = 1, 2.
\end{aligned}$$

The proof is an application of the energy-Casimir method. This result is reminiscent of stability results in [7], where it was shown that long-axis descent of a bottom-heavy prolate spheroid is stable provided the CG is sufficiently low (i.e., $\gamma > 0$ is large enough). A more sophisticated control algorithm, such as the method of controlled Lagrangians, which shapes not only the potential energy but also the kinetic energy and the Hamiltonian structure may provide a broader range of parameters for stability.

6 Conclusions

We have presented two noncanonical, Hamiltonian models for a neutrally buoyant underwater vehicle with a servo-actuated internal mass. These models may aid analysis and control design for other types of vehicles, as well, including spacecraft and atmospheric re-entry vehicles. For example, it has been shown that two servo-actuated masses can provide nutation damping for a spacecraft [13]. Moving masses have also been suggested for re-entry vehicle targeting maneuvers [11, 12]. Because the dynamic models are relatively high-dimensional, past analysis and control design has relied on linear techniques. However, by recognizing the inherent dynamic structure and exploiting it, one may obtain more global results.

Ongoing research seeks to describe the models presented here in terms of reduction theory [9]. Other work focuses on developing practical control strategies for internally actuated vehicles.

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