Lectures 6 & 7: Stability of Equilibria for Time-Varying Systems

Time-varying systems arise quite often in feedback control. Examples abound for which system parameters vary with time. For an aircraft, for example, mass can decrease dramatically as fuel is burned and air density changes throughout the various phases of flight. For spacecraft, parameter values can vary as the vehicle passes in and out of eclipse or, more gradually, as components wear. Another context in which time-varying systems arise is in *trajectory tracking.* In this problem, one attempts to design a control law which drives a given (possibly time-invariant) system along a particular path or trajectory. For example, consider the time-invariant system

$$\dot{\boldsymbol{y}} = \boldsymbol{g}(\boldsymbol{y}, \boldsymbol{u}) \tag{1}$$

and suppose one has designed a feedback control law $\boldsymbol{u}(\boldsymbol{y}, \boldsymbol{y}_{\mathrm{d}}(t))$ to force $\boldsymbol{y}$ to converge to some trajectory of interest $\boldsymbol{y}_{\mathrm{d}}(t)$. We assume that $\boldsymbol{y}_{\mathrm{d}}(t)$ is a feasible trajectory for the feedback controlled system, that is,

$$\dot{\boldsymbol{y}}_{\mathrm{d}} = \boldsymbol{g}(\boldsymbol{y}_{\mathrm{d}}, \boldsymbol{u}(\boldsymbol{y}_{\mathrm{d}}, \boldsymbol{y}_{\mathrm{d}})).$$

We would like to verify that the desired trajectory $\boldsymbol{y}_{\mathrm{d}}(t)$ is asymptotically stable for the closed-loop system

$$\dot{\boldsymbol{y}} = \boldsymbol{g}(\boldsymbol{y}, \boldsymbol{u}(\boldsymbol{y}, \boldsymbol{y}_{\mathrm{d}}(t))). \tag{2}$$

However, we have not introduced any machinery for studying stability of trajectories; we have only studied stability of equilibria.

In fact, stability of a *trajectory* can be rephrased in terms of stability of an *equilibrium* for a nonautonomous system. Let

$$\boldsymbol{x} = \boldsymbol{y} - \boldsymbol{y}_{\mathrm{d}}.$$

Differentiating, we find that

$$\begin{aligned} \dot{\boldsymbol{x}} &= \dot{\boldsymbol{y}} - \dot{\boldsymbol{y}}_{\mathrm{d}} \\ &= \boldsymbol{g}(\boldsymbol{y}, \boldsymbol{u}(\boldsymbol{y}, \boldsymbol{y}_{\mathrm{d}}(t))) - \boldsymbol{g}(\boldsymbol{y}_{\mathrm{d}}(t), \boldsymbol{u}(\boldsymbol{y}_{\mathrm{d}}(t), \boldsymbol{y}_{\mathrm{d}}(t))) \\ &= \boldsymbol{g}\left((\boldsymbol{x} + \boldsymbol{y}_{\mathrm{d}}(t)), \boldsymbol{u}((\boldsymbol{x} + \boldsymbol{y}_{\mathrm{d}}(t)), \boldsymbol{y}_{\mathrm{d}}(t))\right) - \boldsymbol{g}(\boldsymbol{y}_{\mathrm{d}}(t), \boldsymbol{u}(\boldsymbol{y}_{\mathrm{d}}(t), \boldsymbol{y}_{\mathrm{d}}(t))) \\ &=: \boldsymbol{f}(\boldsymbol{x}, t). \end{aligned}$$

Note that the origin $\boldsymbol{x} = \boldsymbol{0}$ is an equilibrium, that is,

$$\boldsymbol{x}(t_0) = \boldsymbol{0} \qquad \Rightarrow \qquad \boldsymbol{x}(t) = \boldsymbol{0} \qquad \forall\, t \geq t_0.$$

If this equilibrium is asymptotically stable for the system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t)$, then the desired trajectory is asymptotically stable for the system (2).

With this example as motivation, we now consider systems of the form

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t) \tag{3}$$

where $\boldsymbol{f}$ is piecewise continuous in t and locally Lipschitz in $\boldsymbol{x}$ in a domain $D \times [0, \infty)$ where $D \subset \mathbb{R}^n$ contains the origin $\boldsymbol{x} = \boldsymbol{0}$. We assume that the origin is an equilibrium point at $t = 0$.

Definition 4.4 [1]: The equilibrium point $\boldsymbol{x} = \boldsymbol{0}$ for the system (3) is

- **stable** if, for each $\epsilon > 0$, there exists $\delta(\epsilon, t_0) > 0$ such that

$$\|\boldsymbol{x}(t_0)\| \leq \delta \qquad \Rightarrow \qquad \|\boldsymbol{x}(t)\| \leq \epsilon \qquad \forall\, t \geq t_0 \geq 0.$$

- **uniformly stable** if it is stable with δ independent of t_0.

- **unstable** if it is not stable.
- **asymptotically stable** if it is stable and there exists $c(t_0) > 0$ such that $\boldsymbol{x}(t) \to 0$ as $t \to \infty$ for all $\|\boldsymbol{x}(t_0)\| < c(t_0)$.
- **uniformly asymptotically stable** if it is uniformly stable, asymptotically stable with c independent of t_0, and convergence of $\boldsymbol{x}(t)$ to $\mathbf{0}$ is uniform in t_0. That is, for every $\epsilon > 0$, there exists $T(\epsilon) > 0$ such that
$$\|\boldsymbol{x}(t)\| \leq \epsilon \qquad \forall\, t \geq t_0 + T(\epsilon) \text{ and } \|\boldsymbol{x}(t_0)\| < c.$$
- **globally uniformly asymptotically stable** if it is uniformly asymptotically stable and ϵ and c can be chosen arbitrarily large.

For examples of equilibria which are stable, but not uniformly stable, and so on, see [1]. Generally, one requires stability to be uniform in order to ensure that the region of attraction does not shrink with time, that convergence rates do not vary with initial time t_0, etc.

Class $\mathcal{K}$ and $\mathcal{KL}$ functions. The stability definitions above can be stated more concisely in terms of special classes of functions known as class $\mathcal{K}$ and $\mathcal{KL}$ functions.

Definition 4.2 [1]: A continuous function $\alpha : [0, a) \to \mathbb{R}^+$ belongs to **class** $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It belongs to **class** $\mathcal{K}_\infty$ if $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$.

Definition 4.3 [1]: A continuous function $\beta : [0, a) \times \mathbb{R}^+ \to \mathbb{R}^+$ belongs to **class** $\mathcal{KL}$ if

- $\beta(r, s)$ is class $\mathcal{K}$ with respect to r for each fixed s, and
- $\beta(r, s)$ is decreasing in s for each fixed r and $\beta(r, s) \to 0$ as $s \to \infty$.

Examples:

- Class $\mathcal{K}$: $\alpha(r) = \tan(r)$ for $r \in [0, \frac{\pi}{2})$; $\alpha(r) = \tanh(r)$ for $r \in [0, \infty)$
- Class $\mathcal{K}_\infty$: $\alpha(r) = kr$ for $k > 0$; $\alpha(r) = r^c$ for $c > 0$; $\alpha(r) = \sinh(r)$
- Class $\mathcal{KL}$: $\beta(r, s) = kre^{-s}$ for $k > 0$.

Lemma 4.2 [1]: Suppose α_1 and α_2 are class $\mathcal{K}$ on $[0, a)$, α_3 and α_4 are class $\mathcal{K}_\infty$, and β is class $\mathcal{KL}$.

- α_1^{-1} is class $\mathcal{K}$ on $[0, \alpha_1(a))$.
- α_3^{-1} is class $\mathcal{K}_\infty$.
- $\alpha_1 \circ \alpha_2$ is class $\mathcal{K}$ on $[0, a)$.
- $\alpha_3 \circ \alpha_4$ is class $\mathcal{K}_\infty$.
- $\alpha_1(\beta(\alpha_2(r), s))$ is class $\mathcal{KL}$.

Aside: Given a locally Lipschitz, class $\mathcal{K}$ function α on $[0, a)$, one can construct a class $\mathcal{KL}$ function on $[0, a) \times [0, \infty)$ by solving the ODE
$$\dot{z} = -\alpha(z), \qquad z(t_0) = z_0$$
for $z_0 \in [0, a)$. For example, consider the class $\mathcal{K}_\infty$ function $\alpha(r) = kr$ where $k > 0$. Solving
$$\dot{z} = -kz, \qquad z(t_0) = z_0$$

gives

$$z(t) = z_0 e^{-k(t-t_0)}.$$

The function $\sigma(r,s) = re^{-ks}$ is class $\mathcal{KL}$. $\square$

Suppose $V(\boldsymbol{x})$ is continuous and positive definite in a ball $B_r \subset \mathbb{R}^n$. Then there exist locally Lipschitz, class $\mathcal{K}$ functions α_1 and α_2 on $[0,r)$ such that

$$\alpha_1(\|\boldsymbol{x}\|) \leq V(\boldsymbol{x}) \leq \alpha_2(\|\boldsymbol{x}\|)$$

for all $\boldsymbol{x} \in B_r$. Moreover, if $V(\boldsymbol{x})$ is defined on all of $\mathbb{R}^n$ and radially unbounded, then α_1 and α_2 can be chosen of class $\mathcal{K}_\infty$. These observations suggest that the stability definitions can be re-worded in terms of class $\mathcal{K}$ and $\mathcal{KL}$ functions.

Lemma 4.5 [1]. The equilibrium point $\boldsymbol{x} = \boldsymbol{0}$ for the system (3) is

- *uniformly stable* if and only if there exist a class $\mathcal{K}$ function $\alpha(\cdot)$ and a constant $c > 0$, independent of t_0, such that
$$\|\boldsymbol{x}(t)\| \leq \alpha(\|\boldsymbol{x}(t_0)\|), \qquad \forall\, t \geq t_0 \geq 0 \text{ and } \|\boldsymbol{x}(t_0)\| < c. \tag{4}$$
- *uniformly asymptotically stable* if and only if there exist a class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$ and a constant $c > 0$, independent of t_0, such that
$$\|\boldsymbol{x}(t)\| \leq \beta(\|\boldsymbol{x}(t_0)\|, t - t_0), \qquad \forall\, t \geq t_0 \geq 0 \text{ and } \|\boldsymbol{x}(t_0)\| < c. \tag{5}$$
- *globally uniformly asymptotically stable* if it is uniformly asymptotically stable and (5) holds for any $\boldsymbol{x}(t_0)$.

Theorem 4.8 [1] (Lyapunov's Stability Theorem for Time-Varying Systems). Consider the system (3) with an equilibrium at the origin. Let $V : D \times [0,\infty) \to R$ be a continuously differentiable function such that

$$W_1(\boldsymbol{x}) \leq V(\boldsymbol{x},t) \leq W_2(\boldsymbol{x})$$
$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial \boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x},t) \leq -W_3(\boldsymbol{x})$$

for all $t \geq 0$ and $\boldsymbol{x} \in D$ where W_1 and W_2 are continuous and positive definite. If $W_3 = 0$ in D, then $\boldsymbol{x} = \boldsymbol{0}$ is *uniformly stable.* If W_3 is continuous and positive definite in D, then $\boldsymbol{x} = \boldsymbol{0}$ is *uniformly asymptotically stable.* If W_3 is continuous and positive definite, all assumptions hold globally, and $W_1(\boldsymbol{x})$ is radially unbounded, then $\boldsymbol{x} = \boldsymbol{0}$ is *globally uniformly asymptotically stable.* $\square$

Proof. The proof, which is taken from [1], is similar to the autonomous case but with minor adjustments for the time-varying dynamics. We start with the case $W_3(\boldsymbol{x}) = 0$. Choose $r > 0$ small enough that $B_r \subset D$ and choose c such that

$$0 < c < \alpha = \min_{\|\boldsymbol{x}\|=r} W_1(\boldsymbol{x}).$$

Then

$$S_{\text{outer}} = \{\boldsymbol{x} \in B_r \mid W_1(\boldsymbol{x}) \leq c\}$$

is in the interior of B_r. Let

$$\Omega_{t,c} = \{\boldsymbol{x} \in B_r \mid V(\boldsymbol{x},t) \leq c\}$$

and note that $\Omega_{t,c} \subseteq S_{\text{outer}}$ because $V(\boldsymbol{x},t) \leq c$ implies that $W_1(\boldsymbol{x}) \leq c$. On the other hand,

$$S_{\text{inner}} = \{\boldsymbol{x} \in B_r \mid W_2(\boldsymbol{x}) \leq c\} \subseteq \Omega_{t,c}$$

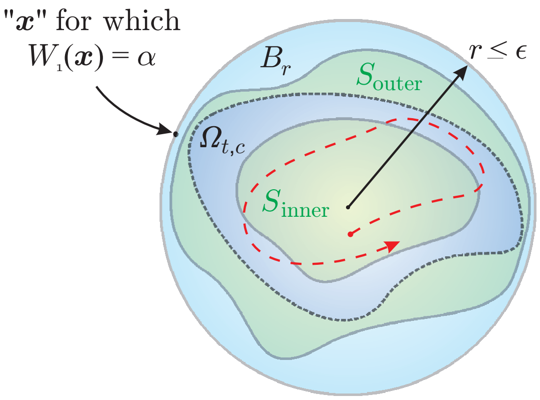

Figure 1: Cartoon for the proof of Lyapunov's Stability Theorem (Nonautonomous case).

because $W_2(\boldsymbol{x}) \leq c$ implies that $V(\boldsymbol{x}, t) \leq c$ (i.e., $\boldsymbol{x} \in S_{\text{inner}}$ implies $\boldsymbol{x} \in \Omega_{t,c}$). To recap,

$$S_{\text{inner}} \subseteq \Omega_{t,c} \subseteq S_{\text{outer}} \subset B_r \subset D$$

for all $t \geq 0$. See Figure 1.

Now, since $\dot{V}(\boldsymbol{x}, t) \leq 0$ in D, any trajectory $\boldsymbol{x}(t)$ for which $\boldsymbol{x}(t_0) \in \Omega_{t_0,c}$ remains in $\Omega_{t,c}$ for all $t \geq t_0$. (See Theorem 3.3 in [1].) Thus, any trajectory starting in the (time-invariant) set S_{inner} remains in $\Omega_{t,c} \subset S_{\text{outer}}$ for all $t \geq t_0$. The trajectory is therefore bounded and exists for all $t \geq t_0$.

We know that there exist locally Lipschitz class $\mathcal{K}$ functions α_1 and α_2 such that

$$\alpha_1(\|\boldsymbol{x}\|) \leq W_1(\boldsymbol{x}) \leq V(\boldsymbol{x}, t) \leq W_2(\boldsymbol{x}) \leq \alpha_2(\|\boldsymbol{x}\|).$$

It follows that

$$\begin{aligned} \|\boldsymbol{x}(t)\| &\leq \alpha_1^{-1}(V(\boldsymbol{x}, t)) \\ &\leq \alpha_1^{-1}(V(\boldsymbol{x}(t_0), t_0)) \qquad \text{(since } \dot{V} \leq 0) \\ &\leq \alpha_1^{-1}(\alpha_2(\|\boldsymbol{x}(t_0)\|)). \end{aligned}$$

Because $\alpha_1^{-1} \circ \alpha_2$ is class $\mathcal{K}$, it follows that the origin is uniformly stable.

To conclude, in the case where W_3 is positive definite in D, that any trajectory starting in $\Omega_{t_0,c}$ converges to the origin, one may use the Lyapunov function to compute an explicit lower bound on the convergence rate. Since W_3 is positive definite, there exists a locally Lipschitz class $\mathcal{K}$ function α_3 such that

$$W_3(\boldsymbol{x}) \geq \alpha_3(\|\boldsymbol{x}\|).$$

We therefore have

$$\begin{gathered} \alpha_1(\|\boldsymbol{x}\|) \leq V(\boldsymbol{x}, t) \leq \alpha_2(\|\boldsymbol{x}\|) \\ \dot{V} \leq -\alpha_3(\|\boldsymbol{x}\|) \end{gathered}$$

and it follows that

$$\dot{V} \leq -\alpha_3(\|\boldsymbol{x}\|) \leq -\alpha_3(\alpha_2^{-1}(V)) =: -\alpha(V)$$

where α is class $\mathcal{K}$ and locally Lipschitz. By the comparison lemma, one finds that

$$\begin{aligned} V(\boldsymbol{x}, t) &\leq \sigma(V(\boldsymbol{x}(t_0), t_0), t - t_0) \\ &\leq \sigma(\alpha_2(\|\boldsymbol{x}(t_0)\|), t - t_0) \end{aligned}$$

where $\sigma(r,s)$ is class $\mathcal{KL}$. It follows that

$$\begin{aligned}
\|\boldsymbol{x}(t)\| &\leq \alpha_1^{-1}\left(V(\boldsymbol{x}(t),t)\right) \\
&\leq \alpha_1^{-1}\left(\sigma(\alpha_2(\|\boldsymbol{x}(t_0)\|),t-t_0)\right) \\
&=: \beta(\|\boldsymbol{x}(t_0)\|,t-t_0)
\end{aligned}$$

where β is class $\mathcal{KL}$. This implies that $\boldsymbol{x}=\boldsymbol{0}$ is uniformly asymptotically stable. □

A time-varying function $V(\boldsymbol{x},t)$ which satisfies $W_1(\boldsymbol{x}) \leq V(\boldsymbol{x},t)$, where $W_1(\boldsymbol{x})$ is positive definite, is called *positive definite*. A function $V(\boldsymbol{x},t)$ which satisfies $V(t,\boldsymbol{x}) \leq W_2(\boldsymbol{x})$, where $W_2(\boldsymbol{x})$ is positive definite, is called *decrescent*. In the time-invariant case, positive definiteness and decrescence are equivalent, so there is no need to require the latter property explicitly. In the time-varying case, decrescence plays an important part in allowing one to compute the explicit convergence rate of the Lyapunov function V.

Example: The origin of the scalar system

$$\dot{x} = -(1+\sin^2(t))x^3$$

is globally uniformly asymptotically stable. To see this, let $V(x)=\frac{1}{2}x^2$. Then

$$\dot{V} = -(1+\sin^2(t))x^4 \leq -x^4.$$

Thus, letting $W_1(x)=W_2(x)=\frac{1}{2}x^2$ and letting $W_3(x)=x^4$, we see that all conditions of the preceding theorem for global uniform asymptotic stability are satisfied.

In the special case that the class $\mathcal{K}$ functions in the sketch of the proof take the form $\alpha_i(r)=k_i r^c$ for $k_i>0$ and $c>0$, we can make the even stronger conclusion that the equilibrium is uniformly *exponentially* stable (or globally uniformly exponentially stable). In this case, we have

$$k_1\|\boldsymbol{x}\|^c \leq V(\boldsymbol{x},t) \leq k_2\|\boldsymbol{x}\|^c \tag{6}$$

$$\dot{V}(\boldsymbol{x},t) \leq -k_3\|\boldsymbol{x}\|^c \tag{7}$$

which implies that

$$\dot{V}(\boldsymbol{x},t) \leq -\frac{k_3}{k_2}V(\boldsymbol{x},t).$$

By the comparison lemma,

$$V(\boldsymbol{x},t) \leq V(\boldsymbol{x}(t_0),t-t_0)e^{-\frac{k_3}{k_2}(t-t_0)}.$$

And since $k_1\|\boldsymbol{x}\|^c \leq V(\boldsymbol{x},t)$, we find that

$$\begin{aligned}
\|\boldsymbol{x}(t)\| &\leq \left(\frac{1}{k_1}V(\boldsymbol{x}(t_0),t-t_0)e^{-\frac{k_3}{k_2}(t-t_0)}\right)^{\frac{1}{c}} \\
&\leq \left(\frac{k_2}{k_1}\|\boldsymbol{x}(t_0)\|^c e^{-\frac{k_3}{k_2}(t-t_0)}\right)^{\frac{1}{c}} \\
&= \|\boldsymbol{x}(t_0)\|\left(\frac{k_2}{k_1}e^{-\frac{k_3}{k_2}(t-t_0)}\right)^{\frac{1}{c}}.
\end{aligned}$$

The convergence rate of the Lyapunov function, and therefore the state, is governed by the exponentially decaying envelope

$$\left(\frac{k_2}{k_1}\right)^{\frac{1}{c}} e^{-\frac{k_3}{ck_2}(t-t_0)}.$$

Analysis using Barbalat's Lemma. Consider the preceding theorem on stability of nonautonomous systems and suppose it is only known that $\dot{V}(\boldsymbol{x}(t),t) \leq 0$. Unlike in the time-invariant case, we can not apply Lasalle's invariance principle. To do so would require defining a compact, positively invariant set Ω and a set $E = \{\boldsymbol{x} \in \Omega \mid \dot{V} = 0\}$. Because the system is time-varying, however, it is not clear how to define the set Ω or the set E. (Recall from the proof of the previous stability theorem that the set $\Omega_{t,c}$ is *time-varying.*)

We would like to develop a tool which is *like* Lasalle's principle in the sense that it gives conditions under which $\dot{V} \rightarrow 0$ and V converges to a constant value. We might then conclude that trajectories converge to a set within which $\dot{V} = 0$. This would tell us something about the behavior of trajectories, although it would not tell us as much as Lasalle's principle which says, further, that trajectories converge to the *largest invariant set* contained in the set where $\dot{V} = 0$.

To illustrate some of the difficulties, consider the following comments based on the discussion in [2]. Consider a differentiable function $f(t)$.

1. If $\dot{f} \rightarrow 0$ as $t \rightarrow \infty$, it does *not* follow that f converges to a limit. As an example,

$$f(t) = \sin(\log t) \qquad \Rightarrow \qquad \dot{f}(t) = \frac{\cos(\log t)}{t} \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty.$$

2. If f converges to a limit as $t \rightarrow \infty$, it does *not* follow that $\dot{f} \rightarrow 0$. As an example,

$$f(t) = e^{-t}\sin^2(e^{2t}) \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty,$$

   however

$$\dot{f}(t) = 2e^{t}\sin(2e^{2t}) - e^{-t}\sin^2(e^{2t})$$

   does not tend to zero.

3. If $\dot{f} \leq 0$ and f is lower-bounded , then f converges to a limit as $t \rightarrow \infty$. Per the previous observation, however, it is not necessarily true that $\dot{f} \rightarrow 0$.

Barbalat's Lemma. If the differentiable function $f(t)$ converges to a finite limit as $t \rightarrow \infty$ and if $\dot{f}$ is uniformly continuous, then $\dot{f} \rightarrow 0$ as $t \rightarrow \infty$.[1]

A function $f(t)$ is *continuous* on $[0,\infty)$ if for every $t_1 \geq 0$ and every $\epsilon > 0$ there exists a $\delta(t_1,\epsilon) > 0$ such that

$$|t - t_1| < \delta \qquad \Rightarrow \qquad |f(t) - f(t_1)| < \epsilon. \tag{8}$$

The function is called *uniformly continuous* on $[0,\infty)$ if δ does not depend on t_1. That is, a function is uniformly continuous if, given $\epsilon > 0$, the same $\delta(t_1,\epsilon)$ satisfies (8) at any time $t_1 \geq 0$. A sufficient condition for a function to be uniformly continuous is that its derivative be bounded.

Corollary. If a scalar function $V(\boldsymbol{x},t)$ satisfies the conditions

- $V(\boldsymbol{x},t)$ is lower bounded,
- $\dot{V}(\boldsymbol{x},t) \leq 0$, and
- $\dot{V}(\boldsymbol{x},t)$ is uniformly continuous in time,

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[1] See Lemma 8.2 in [1].

then $\dot{V}(\boldsymbol{x}, t) \to 0$ as $t \to \infty$.

The corollary says that trajectories converge to a set E within which $\dot{V} = 0$. This conclusion is much weaker than Lasalle's invariance principle; we can not conclude that trajectories converge to the largest invariant set contained in E.

Example. (From [2].) Consider the adaptive control system

$$\begin{aligned} \dot{e} &= -e + \theta d \\ \dot{\theta} &= -ed \end{aligned}$$

where e is the tracking error, θ is an adaptive control parameter, and d is a time-varying disturbance input. Consider the quadratic function

$$V = \frac{1}{2}e^2 + \frac{1}{2}\theta^2.$$

Differentiating,

$$\dot{V} = e(-e + \theta d) + \theta(-ed) = -e^2 \leq 0$$

Because of the disturbance signal $d(t)$, the system is time-varying and we may not use Lasalle's invariance principle. To verify uniform continuity of $\dot{V}$, we check that

$$\ddot{V} = -2e(-e + \theta d)$$

is bounded. First note that, because V is positive definite and nonincreasing, e and θ are bounded:

$$\left\| \begin{pmatrix} e \\ \theta \end{pmatrix} \right\| \leq 2\sqrt{V(0)}.$$

Assuming that the reference signal d is also bounded, which is not unreasonable, it follows that $\ddot{V}$ is bounded and therefore that $\dot{V}$ is uniformly continuous. It then follows from the previous corollary to Barbalat's lemma that $e \to 0$ as $t \to \infty$. Note that we can conclude nothing more about the behavior of θ, except that its value remains bounded.

Linear time-varying systems. Before discussing the nonautonomous extension of Lyapunov's indirect method, we make a few observations about stability of linear time-varying systems.

First of all, stability of linear, time-varying systems is generally more difficult to gauge than stability of linear, time-invariant systems. One cannot simply check the eigenvalues of the state matrix to determine stability, in general. Khalil gives the following example [1]. Define

$$\boldsymbol{A}(t) = \begin{pmatrix} -1 + 1.5\cos^2 t & 1 - 1.5\cos^2 t \\ -1 - 1.5\cos^2 t & -1 + 1.5\sin^2 t \end{pmatrix}.$$

The eigenvalues of $\boldsymbol{A}(t)$ are $\frac{1}{4}(-1 \pm j\sqrt{7})$. Thus the eigenvalues are constant and have negative real part. However, the trajectory $\boldsymbol{x}(t)$ corresponding to

$$\dot{\boldsymbol{x}} = \boldsymbol{A}(t)\boldsymbol{x}, \qquad \boldsymbol{x}(0) = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$$

for $\alpha \in \mathbb{R}$ is

$$\boldsymbol{x}(t) = \alpha \begin{pmatrix} e^{0.5t}\cos t \\ -e^{0.5t}\sin t \end{pmatrix}$$

which diverges as $t \to \infty$: $\|\boldsymbol{x}(t)\| = |\alpha|e^{0.5t}$. To determine stability of a time-varying linear system, one must generally examine the properties of the state transition matrix for the system. This equates to solving the system dynamics explicitly. This is often unappealing, but it can certainly be done.

Alternatively, one can use Lyapunov theory to prove global uniform asymptotic stability for a linear time-varying system. Consider the linear, time-varying system

$$\dot{\boldsymbol{x}} = \boldsymbol{A}(t)\boldsymbol{x} \tag{9}$$

where the components of $\boldsymbol{A}(t)$ are continuous. Suppose there is a continuously differentiable, symmetric matrix $\boldsymbol{P}(t)$ which is positive definite ($\boldsymbol{q}^T\boldsymbol{P}(t)\boldsymbol{q} \geq c_1\|\boldsymbol{q}\|^2 \geq 0$ for some $c_1 > 0$), decrescent ($\boldsymbol{q}^T\boldsymbol{P}(t)\boldsymbol{q} \leq c_2\|\boldsymbol{q}\|^2$ for some $c_2 > 0$), and satisfies the time-varying Lyapunov equation

$$-\dot{\boldsymbol{P}}(t) = \boldsymbol{P}(t)\boldsymbol{A}(t) + \boldsymbol{A}^T(t)\boldsymbol{P}(t) + \boldsymbol{Q}(t) \tag{10}$$

where $\boldsymbol{Q}(t)$ is positive definite. Then the origin is globally exponentially stable for the system (9). To prove this, let $V(\boldsymbol{x}, t) = \boldsymbol{x}^T\boldsymbol{P}(t)\boldsymbol{x}$. Then

$$c_1\|\boldsymbol{x}\|^2 \leq V(\boldsymbol{x}, t) \leq c_2\|\boldsymbol{x}\|^2$$

and

$$\dot{V} = -\boldsymbol{x}^T\boldsymbol{Q}(t)\boldsymbol{x} \leq -c_3\|\boldsymbol{x}\|^2$$

where $c_3 > 0$. The origin is therefore exponentially stable.

We have seen that one may prove global exponential stability of a linear, time-varying system by constructing a Lyapunov function. (Asymptotic and exponential stability are equivalent for linear systems.) Conversely, it is also true that global exponential stability of the origin implies the existence of a Lyapunov function. The following theorem is known as a "converse theorem" for an asymptotically (exponentially) stable linear system. As noted at the end of this lecture, there are similar converse theorems for stable nonlinear systems.

Theorem 4.12 [1]. Suppose the linear time-varying system

$$\dot{\boldsymbol{x}} = \boldsymbol{A}(t)\boldsymbol{x}$$

has a globally uniformly asymptotically stable equilibrium at the origin, where $\boldsymbol{A}(t)$ is continuous and bounded. Let $\boldsymbol{Q}(t)$ be any continuous, positive definite, decrescent, symmetric matrix. Then there exists a continuously differentiable, positive definite, decrescent, symmetric matrix $\boldsymbol{P}(t)$ which satisfies the time-varying Lyapunov equation (10). It follows that $V(\boldsymbol{x}, t) = \boldsymbol{x}^T\boldsymbol{P}(t)\boldsymbol{x}$ is a Lyapunov function for the equilibrium.

Ignoring the difficulties associated with checking stability of linear, time-varying systems, we can state the following non-autonomous extension of Lyapunov's indirect method.

Theorem 4.13 [1] (Lyapunov's Indirect Method for Time-Varying Systems). Suppose $\boldsymbol{x} = \boldsymbol{0}$ is an equilibrium for the nonlinear, time-varying system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t)$$

where $\boldsymbol{f} : [0, \infty) \times B_r \to \mathbb{R}^n$ is continuously differentiable and the Jacobian matrix $\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}$ is bounded and Lipschitz on B_r, uniformly in t. The origin is exponentially stable for the nonlinear system if it is exponentially stable for the linear system

$$\dot{\boldsymbol{x}} = \left(\left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}=\boldsymbol{0}}\right)\boldsymbol{x} =: \boldsymbol{A}(t)\boldsymbol{x}.$$

The proof involves the previous converse theorem concerning existence of a Lyapunov function for a stable linear system, and it is similar to the proof of Lyapunov's indirect method for the time-invariant case. Essentially, one uses the Lyapunov function for the linear system as a candidate Lyapunov function for the nonlinear system and shows that there is a sufficiently small neighborhood around the origin in which the function is strictly decreasing. See [1].

Converse Theorems. The utility of Lyapunov functions extends well beyond proving stability or asymptotic stability. As we have already seen, for example, they can be used to obtain an estimate of the region of attraction for an asymptotically stable equilibrium. In the next lecture, we will define the notion of input-to-state stability, a way of characterizing a system's response to inputs (either controls or disturbances). As we shall see, input-to-state stability is closely related to Lyapunov stability theory.

Given that Lyapunov functions have uses beyond simply assessing stability, one might naturally wonder the following: Given a nonlinear system with an asymptotically stable equilibrium, does there exist a Lyapunov function which proves asymptotic stability (i.e., a positive definite function whose rate is negative definite)? As shown in Section 4.7 of [1], the answer is a qualified "yes." Khalil states and proves four theorems in Section 4.7, which are paraphrased below, without detailing the assumptions and conditions. (See Section 4.7 of [1] for the precise statements.)

Theorem 4.14 [1]: If $\boldsymbol{f}(\boldsymbol{x},t)$ is continuously differentiable, then exponential stability of the origin of the nonlinear system implies the existence of a positive definite, decrescent Lyapunov function, whose gradient satisfies a linear growth bound in $\|\boldsymbol{x}\|$ and whose rate is negative definite. Moreover, the bounding functions $W_i(\boldsymbol{x})$ can be chosen quadratic in $\boldsymbol{x}$ for $i \in \{1,2,3\}$. If the system is autonomous, then the Lyapunov function can be chosen independently of time.

Theorem 4.15 [1]: If $\boldsymbol{f}(\boldsymbol{x},t)$ is continuously differentiable, then exponential stability of the origin for the linear approximation is necessary and sufficient for exponential stability of the origin for the nonlinear dynamics.

Corollary 4.3 [1]: For a time-invariant system, the origin is exponentially stable for the nonlinear dynamics if and only if the linear state matrix is Hurwitz.

Theorem 4.16 [1]: If $\boldsymbol{f}(\boldsymbol{x},t)$ is continuously differentiable, then asymptotic stability of the origin implies the existence of a positive definite, decrescent Lyapunov function whose gradient satisfies a class $\mathcal{K}$ growth bound in $\|\boldsymbol{x}\|$ and whose rate is negative definite. If the system is autonomous, then the Lyapunov function can be chosen independently of time. (This is a more general version of Theorem 4.14.)

Theorem 4.17 [1]: For an autonomous system, asymptotic stability of the origin implies the existence of a Lyapunov function which is positive definite and strictly decreasing in the entire region of attraction R_{A} and such that

$$V(\boldsymbol{x}) \to \infty \qquad \Rightarrow \qquad \boldsymbol{x} \to \delta R_{\mathrm{A}},$$

where δR_{A} denotes the boundary of R_{A}. If $R_{\mathrm{A}} = \mathbb{R}^n$, then $V(\boldsymbol{x})$ is radially unbounded.

References

[1] H. K. Khalil. *Nonlinear Systems.* Prentice-Hall, third edition, 2002.

[2] J.-J. Slotine and W. Li. *Applied Nonlinear Control.* Prentice-Hall, 1991.