Lecture 8: Boundedness & Input-to-State Stability

Boundedness & Ultimate Boundedness. For some systems, it is necessary to relax the notion of stability and seek, instead, only *boundedness* of the state to a particular neighborhood of the origin. Consider, as a motivating example from [1], the following system:

$$\dot{x} = -x + \delta \sin t \qquad x(t_0) = a \tag{1}$$

where $a > \delta > 0$. The solution is

$$x(t) = ae^{-(t-t_0)} + \delta \int_{t_0}^{t} e^{-(t-\tau)} \sin \tau d\tau \,. \tag{2}$$

A bound on the solution can be computed directly:

$$|x(t)| \leq ae^{-(t-t_0)} + \delta \int_{t_0}^{t} e^{-(t-\tau)} d\tau = ae^{-(t-t_0)} + \delta \left[1 - e^{-(t-t_0)}\right] \leq a, \quad \forall\, t \geq t_0. \tag{3}$$

Thus, the solution is bounded for all $t \geq t_0$, uniformly in t_0, i.e. with a bound that is independent of t_0. That is, the solution is *uniformly bounded*. While this bound is valid for all $t \geq t_0$, it does not take into consideration that the term whose coefficient is a decays exponentially with time. In terms of the "steady state" behavior, this bound is overly conservative. In fact, for any number b satisfying $\delta < b < a$, the following is true:

$$|x(t)| \leq b, \qquad \forall\, t \geq t_0 + \ln\left(\frac{a-\delta}{b-\delta}\right) \tag{4}$$

The bound b, which again is independent of t_0, gives a better estimate of the solution after a transient period has passed. In this case, the solution is said to be *uniformly ultimately bounded*, and b is called the ultimate bound. Of course, if the solution is bounded, then it is ultimately bounded, and vice versa. The value of the latter notion is that the ultimate bound may be much smaller than the overall bound.

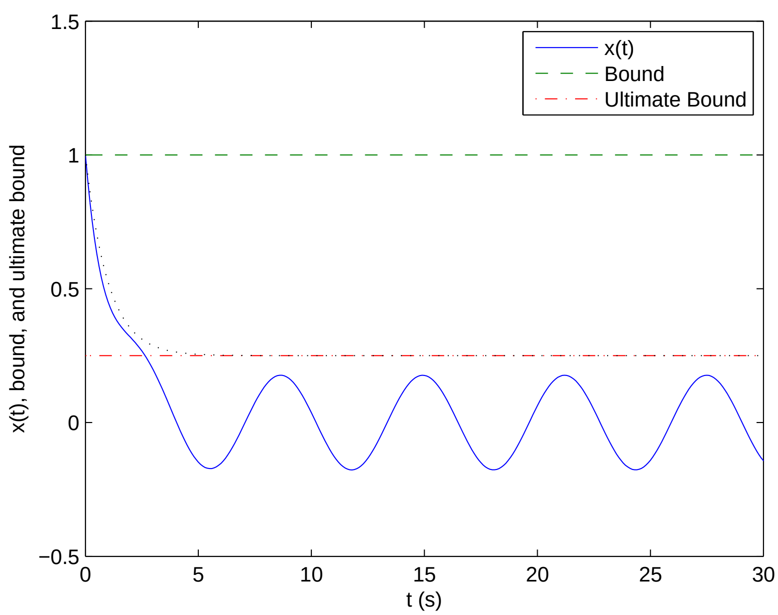

Figure 1: Example of boundedness and ultimate boundedness.

Note that boundedness and ultimate boundedness are properties of a system, rather than of an equilibrium of a system. In fact, the example system above has no equilibrium. Regardless, the properties of system boundedness and ultimate boundedness can be established via Lyapunov analysis in a manner similar to proving stability, or asymptotic stability, of an equilibrium.

Returning to the example above, consider the Lyapunov function candidate $V = \frac{1}{2}x^2$. Its derivative is

$$\dot{V}(t) = x\dot{x} = -x^2 + x\delta \sin t \leq -x^2 + \delta|x| \tag{5}$$

Notice that $\dot{V}(t) < 0$ for all x outside the compact set $\{|x| \leq \delta\}$. For any $c > \frac{1}{2}\delta^2$, solutions starting in the set $\{x \mid V(x) \leq c\}$ will remain there for all future time, since $\dot{V} < 0$ along the boundary $V(x) = c$. Hence, this set is positively invariant and solutions are uniformly bounded. Moreover, for any number $b > 0$ satisfying $\frac{1}{2}\delta^2 < \frac{1}{2}b^2 < c$, $\dot{V}$ will be negative in the set $\{x \mid \frac{1}{2}b^2 \leq V(x) \leq c\}$, which shows that within this set V will decrease monotonically until the solution enters the set $\{x \mid V(x) \leq \frac{1}{2}b^2\}$. (See Figure 2.) From that time on, the solution cannot leave the set $\{x \mid V(x) \leq \frac{1}{2}b^2\}$, because $\dot{V} < 0$ on the boundary $V(x) = \frac{1}{2}b^2$, i.e., whenever $|x| = b$. Therefore, the solution is uniformly ultimately bounded with ultimate bound $|x| \leq b$.

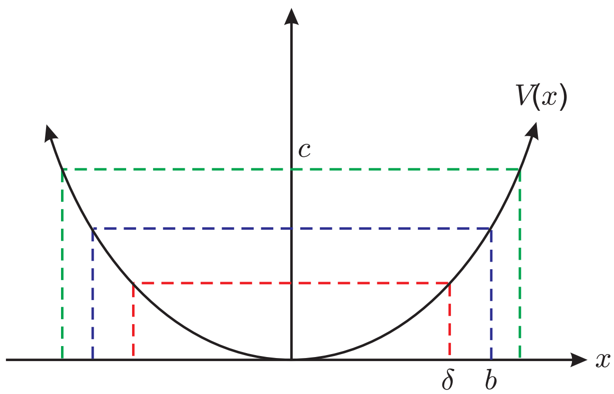

Figure 2: Boundedness and ultimate boundedness from Lyapunov-like analysis.

Definition 4.6 [1] Consider the nonlinear system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t), \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0, \tag{6}$$

where $\boldsymbol{x} \in D \subset \mathbb{R}^n$ is the state of the system, D is an open set containing the origin, and $\boldsymbol{f}(\boldsymbol{x}, t) : D \times \mathbb{R} \to \mathbb{R}^n$ is locally Lipschitz in its arguments. The system trajectories are

- *uniformly bounded* if there exists a positive constant γ, independent of t_0, such that for every $\delta \in (0, \gamma)$, there is $\epsilon = \epsilon(\delta) > 0$, independent of t_0, such that $\|\boldsymbol{x}_0\| \leq \delta$ implies $\|\boldsymbol{x}(t)\| \leq \epsilon$ for all $t \geq t_0$.
- *globally uniformly bounded* if for every $\delta \in (0, \infty)$, there is $\beta = \beta(\delta) > 0$, independent of t_0, such that $\|\boldsymbol{x}_0\| \leq \delta$ implies $\|\boldsymbol{x}(t)\| \leq \beta$ for all $t \geq t_0$.
- *uniformly ultimately bounded with ultimate bound* $b > 0$ if there exists $\gamma > 0$ such that, for every $\delta \in (0, \gamma)$, there exists $T = T(\delta, b) > 0$ such that $\|\boldsymbol{x}_0\| \leq \delta$ implies $\|\boldsymbol{x}(t)\| \leq b$ for all $t \geq t_0 + T$.
- *globally uniformly ultimately bounded*, if for every $\delta \in (0, \infty)$, there exists $T = T(\delta, b) > 0$ such that $\|\boldsymbol{x}_0\| < \delta$ implies $\|\boldsymbol{x}(t)\| < b$ for all $t \geq t_0 + T$.

The main difference between boundedness and stability is that ϵ is not required to be *arbitrarily* small. Recall, in the definition of stability, that one requires the existence of $\delta(\epsilon)$ such that "trajectories starting within δ of the equilibrium stay within ϵ of it" for arbitrarily small ϵ. A system that is bounded does not necessarily possess a Lyapunov stable equilibrium. Likewise, while ultimate boundedness suggests some notion of convergence, it is a considerably weaker property than asymptotic stability.

Despite the fact that boundedness is not equivalent to stability, one may study boundedness using Lyapunov's direct method. Consider a continuously differentiable, positive definite Lyapunov function candidate $V(\boldsymbol{x})$ and a set

$$\Lambda = \{\boldsymbol{x} \mid \epsilon \leq V(\boldsymbol{x}) \leq c\}, \qquad \text{where} \qquad \epsilon \leq c. \tag{7}$$

See Figure 3. Suppose that, along the trajectories of the system (6), we have

$$\dot{V}(\boldsymbol{x}) \leq -W(\boldsymbol{x}) \qquad \text{for all} \qquad \boldsymbol{x} \in \Lambda \qquad \text{and all} \qquad t \geq t_0, \tag{8}$$

where $W(\boldsymbol{x})$ is a continuous, positive definite function. Inequality (8) implies that the sets $\Omega_c = \{\boldsymbol{x} \mid V(\boldsymbol{x}) \leq c\}$ and $\Omega_\epsilon = \{\boldsymbol{x} \mid V(\boldsymbol{x}) \leq \epsilon\}$ are positively invariant since $\dot{V}(t) < 0$ on the boundaries $\partial\Omega_c$ and $\partial\Omega_\epsilon$. Note that, within the set Λ, the conditions of Lyapunov's theorem hold for uniform asymptotic stability of an equilibrium at the origin.[1] It follows that, within the set Λ,

$$\|\boldsymbol{x}(t)\| \leq \beta(\|\boldsymbol{x}(t_0)\|, t - t_0) \tag{9}$$

for some class $\mathcal{KL}$ function β. The function $V(\boldsymbol{x}(t))$ is decreasing in Λ, so every trajectory in Λ enters the set Ω_ϵ in *finite time*. To see that trajectories in Λ enter Ω_ϵ in finite time, first note that $W(\boldsymbol{x})$ has a minimum value in Λ, call it k, because $W(\boldsymbol{x})$ is continuous and Λ is a compact set. Thus,

$$W(\boldsymbol{x}) \geq k \qquad \text{for all} \qquad \boldsymbol{x} \in \Lambda\,. \tag{10}$$

It follows from (8) that

$$\dot{V}(\boldsymbol{x}(t)) \leq -k \qquad \text{for all} \qquad \boldsymbol{x} \in \Lambda \qquad \text{and all} \qquad t \geq t_0\,. \tag{11}$$

Thus

$$V(\boldsymbol{x}(t)) \leq V(\boldsymbol{x}(t_0)) - k(t - t_0) \leq c - k(t - t_0)\,, \tag{12}$$

which shows that $V(\boldsymbol{x}(t))$ decreases to ϵ within the *finite* time interval $[t_0, t_0 + (c - \epsilon)/k]$.

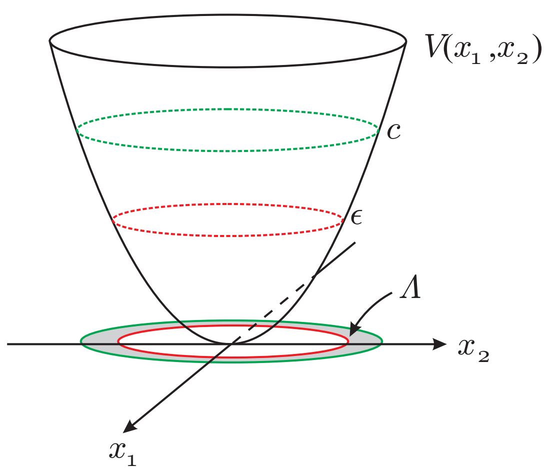

Figure 3: The set Λ.

Of course, the fact that $V(\boldsymbol{x}) \leq \epsilon$ after some finite time doesn't necessarily tell us much about the actual ultimate bound on $\boldsymbol{x}(t)$ unless we know the functional form of $V(\boldsymbol{x})$. The following theorem assumes the existence of *known* class $\mathcal{K}$ functions α_1 and α_2 such that $\alpha_1(\|\boldsymbol{x}\|) \leq V(\boldsymbol{x}, t) \leq \alpha_2(\|\boldsymbol{x}\|)$.

[1] Of course, as we've said, there may not even be an equilibrium at the origin, but the trajectories inside the set Λ don't know this...

Theorem 4.18 [1] Consider the nonlinear system (6). Let $D \subset \mathbb{R}^n$ be a domain that contains the origin and let $V : D \times [0, \infty) \to \mathbb{R}$ be a continuously differentiable function satisfying

$$\alpha_1(\|\boldsymbol{x}\|) \le V(\boldsymbol{x}, t) \le \alpha_2(\|\boldsymbol{x}\|)$$

$$\dot{V}(\boldsymbol{x}, t) \le -W(\boldsymbol{x}) \qquad \text{for all} \qquad \|\boldsymbol{x}\| \ge \mu > 0$$

for all $t \ge t_0$ and all $\boldsymbol{x} \in D$ where α_1 and α_2 are class $\mathcal{K}$ functions and $W(\boldsymbol{x})$ is a continuous, positive definite function. Take $r > 0$ such that $B_r \subset D$ and suppose that $\mu < \alpha_2^{-1}(\alpha_1(r))$. Then for every initial state $\boldsymbol{x}(t_0)$ satisfying $\|\boldsymbol{x}(t_0)\| \le \alpha_2^{-1}(\alpha_1(r))$, there exists a class $\mathcal{KL}$ function β and a time $T(\boldsymbol{x}(t_0), \mu) \ge 0$ such that

$$\|\boldsymbol{x}(t)\| \le \beta(\|\boldsymbol{x}(t_0)\|, t - t_0) \qquad \text{for all} \qquad t_0 \le t \le t_0 + T$$

and

$$\|\boldsymbol{x}(t)\| \le \alpha_1^{-1}(\alpha_2(\mu)) \qquad \text{for all} \qquad t \ge t_0 + T$$

If $D = \mathbb{R}^n$ and α_1 is class $\mathcal{K}_\infty$, then the observations hold globally.

Khalil [1] provides a nice example application showing global uniform ultimate boundedness of Duffing's equation

$$m\ddot{y} + b\dot{y} + ky\left(1 + (ay)^2\right) = A\cos\omega t.$$

Note that, although the system is ultimately bounded, it has no equilibrium at the origin.

Input-to-State Stability. Recall the system

$$\dot{x} = -x + \delta \sin t$$

from our discussion of boundedness and ultimate boundedness. Considering this as the control system

$$\dot{x} = -x + u,$$

with a particular choice of input, one may ask whether boundedness and ultimate boundedness with respect to the input $u = \delta \sin t$ reflects some more general relationship between "well-behaved" inputs and "well-behavedness" of the state. In fact, this control system exhibits a special and very useful property known as *input-to-state stability.* Briefly, an input-to-state stable (ISS) system is one for which the state is uniformly, ultimately bounded under bounded inputs. Moreover, as we shall see in a moment, *any* linear, time-invariant control system with a Hurwitz state matrix is ISS.

Definition 4.7 [1] Consider the system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t)\,, \tag{13}$$

where $\boldsymbol{f}$ is piecewise continuous in t and locally Lipschitz in $\boldsymbol{x}$ and $\boldsymbol{u}$, where $\boldsymbol{x}(t) \in \mathbb{R}^n$, $\boldsymbol{u}(t) \in \mathbb{R}^m$. The components of $\boldsymbol{u}(t)$ may be any piecewise continuous, bounded functions of t. The system (13) is *input-to-state stable* (or *ISS*) if there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ such that for any initial state $\boldsymbol{x}(t_0)$ and any bounded input $\boldsymbol{u}(t)$ the solution $\boldsymbol{x}(t)$ exists for all $t \ge t_0$ and satisfies

$$||\boldsymbol{x}(t)|| \le \beta(||\boldsymbol{x}(t_0)||, t - t_0) + \gamma\Big(\sup_{t_0 \le \tau \le t} ||\boldsymbol{u}(\tau)||\Big)\,. \tag{14}$$

Consider what this definition implies. First, if $\boldsymbol{u} \equiv \boldsymbol{0}$, then

$$||\boldsymbol{x}(t)|| \le \beta(||\boldsymbol{x}(t_0)||, t - t_0). \tag{15}$$

That is, the origin is a globally, uniformly, asymptotically stable equilibrium for the unforced system. So it is already clear that the class of ISS systems is quite special. Furthermore, the ISS property guarantees that for any input such that $\|\boldsymbol{u}(t)\| \leq \bar{u}$, the state of the system will be ultimately bounded by a class $\mathcal{K}$ function of $\bar{u}$, since $\beta \to 0$ as $t \to \infty$.

Consider the case of a stable linear, time-invariant control system

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$

where $\boldsymbol{A}$ is Hurwitz. The solution, for a given initial state $\boldsymbol{x}(t_0)$ and input $\boldsymbol{u}(t)$ is

$$\boldsymbol{x}(t) = \mathrm{e}^{A(t-t_0)}\boldsymbol{x}(t_0) + \int_{t_0}^{t} \mathrm{e}^{A(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau\,.$$

The solution is upper bounded:

$$\begin{aligned} ||\boldsymbol{x}(t)|| &\leq k\mathrm{e}^{-\lambda(t-t_0)}||\boldsymbol{x}(t_0)|| + \int_{t_0}^{t} k\mathrm{e}^{-\lambda(t-\tau)}||\boldsymbol{B}||||\boldsymbol{u}(\tau)||d\tau \\ &\leq k\mathrm{e}^{-\lambda(t-t_0)}||\boldsymbol{x}(t_0)|| + \frac{k}{\lambda}||\boldsymbol{B}|| \sup_{t_0\leq\tau\leq t} ||\boldsymbol{u}(\tau)||\,, \end{aligned}$$

where λ corresponds to the slowest eigenmode. This system is therefore input-to-state stable with

$$\beta(r,s) = kre^{-\lambda s} \qquad \text{and} \qquad \gamma(r) = \frac{k}{\lambda}||\boldsymbol{B}||r.$$

The following theorem gives sufficient conditions under which the more general system (13) is ISS.

Theorem 4.19 [1] Let $V : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^+$ be a continuously differentiable, positive definite function such that

$$\alpha_1(||\boldsymbol{x}||) \leq V(\boldsymbol{x},t) \leq \alpha_2(||\boldsymbol{x}||)$$

$$\dot{V}(t) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial \boldsymbol{x}} f(t,\boldsymbol{x},\boldsymbol{u}) \leq -W(\boldsymbol{x}) \qquad \text{for all} \qquad ||\boldsymbol{x}|| \geq \rho(||\boldsymbol{u}||) > 0$$

and for all $(\boldsymbol{x},\boldsymbol{u},t) \in \mathbb{R}^n \times \mathbb{R}^m \times [0,\infty)$, where α_1 and α_2 are class $\mathcal{K}_\infty$ functions, ρ is a class $\mathcal{K}$ function, and $W(\boldsymbol{x})$ is a continuous, positive definite function on $\mathbb{R}^n$. Then the system (13) is ISS with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$.

The proof of the theorem essentially follows from our previous theorem on global, uniform, ultimate boundedness. The essential point of the theorem is that $\dot{V} < 0$ for $\boldsymbol{x}$ "large enough" relative to $\boldsymbol{u}$. For a given bounded input, no matter how large the bound is, the value of V will decrease if the state grows large enough; the state can not possibly grow unbounded.

A converse Lyapunov theorem for exponentially stable systems, Theorem 4.14 in [1], leads to a more conservative result, but one whose conditions might be more easily checked.

Lemma 4.6 [1] Suppose that $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u},t)$ is continuously differentiable and globally Lipschitz in $(\boldsymbol{x},\boldsymbol{u})$, uniformly in t, and that the *unforced* system in (13) has a globally exponentially stable equilibrium at the origin. Then the forced system is ISS.

Proof (Sketch). Because the origin is globally exponentially stable for the unforced system, there exists a globally valid Lyapunov function $V(\boldsymbol{x},t)$ for the unforced system. Because $\boldsymbol{f}$ is globally Lipschitz in $\boldsymbol{u}$, uniformly in time, there is a constant $L > 0$ for which

$$||\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u},t) - \boldsymbol{f}(\boldsymbol{x},\boldsymbol{0},t)|| \leq L||\boldsymbol{u}|| \qquad \text{for all} \qquad t \geq t_0 \qquad \text{and all} \qquad (\boldsymbol{x},\boldsymbol{u})\,.$$

Computing the derivative of $V(\boldsymbol{x},t)$, one obtains

$$\begin{aligned} \dot{V}(\boldsymbol{x},t) &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial \boldsymbol{x}} f(\boldsymbol{x},\boldsymbol{0},t) + \frac{\partial V}{\partial \boldsymbol{x}}(f(\boldsymbol{x},\boldsymbol{u},t) - f(\boldsymbol{x},\boldsymbol{0},t)) \\ &\leq -c_1||\boldsymbol{x}||^2 + c_2||\boldsymbol{x}||L||\boldsymbol{u}|| \end{aligned}$$

for some positive constants c_1 and c_2. (The linear growth bound on the gradient of V appears in Theorem 4.14.) The rest of the proof involves finding a scale factor $\kappa > 0$ such that $\dot{V} < 0$ for all $\|\boldsymbol{x}\| \geq \kappa\|\boldsymbol{u}\|$, as required in the definition of input-to-state stability.

This lemma imposes two strict conditions to ensure that the system is ISS: a globally Lipschitz vector field $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t)$ and a globally exponentially stable equlibrium at the origin. Although these are only sufficient conditions, their significance is demonstrated by two examples of non-ISS systems for which one or the other condition does not hold.

Example. Consider the system

$$\dot{x} = -3x + (1 + x^2)u.$$

Although the unforced system has a globally exponentially stable equilibrium at $x = 0$, the right hand side is not globally Lipschitz in x and u. To see this, suppose that $u = 1$. As we already saw in an earlier discussion, the right hand side of the system

$$\dot{x} = -3x + (1 + x^2)$$

is not globally Lipschitz. In fact, this system has a finite escape time, so it is certainly not ISS.

Example. The right hand side of the system

$$\dot{x} = -\frac{x}{1 + x^2} + u$$

is globally Lipschitz in x and u, since both partials are globally bounded. Moreover, the origin of the unforced system

$$\dot{x} = -\frac{x}{1 + x^2}$$

is globally asymptotically stable. (Take $V(\boldsymbol{x}) = x^2/2$ as the Lyapunov function.) The equilibrium is also locally exponentially stable. (The linearization is $\dot{x} = -x$.) But the equilibrium is not globally exponentially stable. Moreover, if $u(t) \equiv 1$, then $\dot{x} \geq 1/2$ so that $x(t) \geq x(t_0) + t/2$ for all $t \geq 0$. For this bounded input, the solution is unbounded, so the system cannot be ISS.

The conditions of the lemma are only sufficient, however, and it is possible for a system to be ISS even if these conditions do not hold.

Example. The system

$$\dot{x} = -x^3 + u$$

has a GAS equilibrium at the origin when $u = 0$, but the equilibrium is not GES. Taking $V = x^2/2$, we have

$$\begin{aligned} \dot{V} &= -x^4 + xu \\ &= -(1-\theta)x^4 - \theta x^4 + xu \\ &\leq -(1-\theta)x^4 \qquad \text{for all} \qquad |x| \geq \left(\frac{|u|}{\theta}\right)^{1/3} \qquad \text{with} \qquad 0 < \theta < 1. \end{aligned}$$

Therefore the system is ISS with $\gamma(r) = (r/\theta)^{1/3}$.

Example. The system

$$\dot{x} = -x - 2x^3 + (1 + x^2)u^2$$

has a GES equilibrium at the origin when $u = 0$, but the right hand side is not globally Lipschitz. However, taking $V = x^2/2$, we obtain

$$\dot{V} = -x^2 - 2x^4 + x(1 + x^2)u^2 \leq -x^4 \qquad \text{for all} \qquad |x| > u^2\,.$$

Thus, the system is ISS with $\gamma(r) = r^2$.

A useful application of the concept of input-to-state stability concerns stability analysis of cascaded systems. Consider the system

$$\dot{\boldsymbol{x}}_1 = \boldsymbol{f}_1(\boldsymbol{x}_1, \boldsymbol{x}_2, t), \qquad \boldsymbol{x}_1(t_0) = \boldsymbol{x}_{1_0} \tag{16}$$

$$\dot{\boldsymbol{x}}_2 = \boldsymbol{f}_2(\boldsymbol{x}_2, t), \qquad \boldsymbol{x}_2(t_0) = \boldsymbol{x}_{2_0} \tag{17}$$

where $\boldsymbol{f}_1 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times [0, \infty) \to \mathbb{R}^{n_1}$ and $\boldsymbol{f}_2 : \mathbb{R}^{n_2} \times [0, \infty) \to \mathbb{R}^{n_2}$ are piecewise continuous in t and locally Lipschitz in $\boldsymbol{x} = [\boldsymbol{x}_1^T \ \boldsymbol{x}_2^T]^T$.

Lemma 4.7 [1] If the system (16) is ISS with respect to the input $\boldsymbol{x}_2$ and the origin of (17) is GUAS, then the origin of the combined cascaded system is GUAS.

Thus, if the two subsystems, taken separately, have GUAS equilibria at their respective origins and the first subsystem is ISS with $\boldsymbol{x}_2$ as an input, then the cascade system is GUAS. We will make use of this property when we discuss input/output feedback linearization.

References

[1] H. K. Khalil. *Nonlinear Systems*. Prentice-Hall, third edition, 2002.