Lecture 9: Evans' Rules for Root Loci

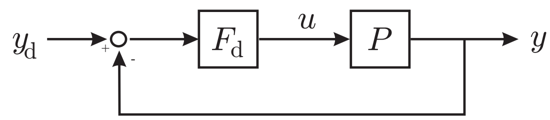

Figure 1: One degree of freedom closed-loop control structure.

Recall that we are considering one degree of freedom feedback as shown in Figure 1. The closed-loop transfer function is

$$\frac{y(s)}{y_d(s)} = H_d(s) = \frac{P(s)F_d(s)}{1 + P(s)F_d(s)}$$

and the closed-loop poles are those values of s for which

$$1 + P(s)F_d(s) = 0.$$

This condition may be rewritten as

$$|P(s)F_d(s)| = 1 \qquad \text{and} \qquad \angle P(s)F_d(s) = (2k+1)\pi \quad k = 0, \pm 1, \pm 2, \ldots$$

and we assume that $P(s)F_d(s)$ can be written in the form

$$P(s)F_d(s) = K\frac{b(s)}{a(s)}.$$

where $b(s)$ has degree m, $a(s)$ has degree $n \geq m$ and where $K > 0$ is a parameter (e.g., a control gain) which may vary. Continuing with our increasingly complicated examples, consider the following.

Example 3. Next consider a system for which

$$P(s)F_d(s) = K\frac{s^2 + 3s + 2}{s^2 + 2s + 4}.$$

The loop gain has two zeros at $z_1 = -1$ and $z_2 = -2$ and two poles at $p_{1,2} = -1 \pm i\sqrt{3}$.

First, determine if any closed-loop poles lie on the real axis. The vectors from the two poles to points on the real axis form an isosceles triangle. The sum of their contribution to $\angle PF_d$ is -2π. Similarly, a pair of complex conjugate zeros would contribute 2π to the angle calculation. Thus, we see that *complex conjugate poles and zeros have no impact on whether a given segment of the real axis is part of the root locus.* The real axis poles and zeros alone determine which portion of the real axis is part of the root locus. In fact, one can easily verify that the following statement is true:

- The portion of the real axis which is part of the root locus lies to the left of an odd number of poles and zeros.

For the present example, we observe that the only part of the real axis which contributes to the root locus lies between the zeros at z_1 and z_2. Recall that for small values of K, the closed-loop poles are close to the poles of the loop gain $P(s)F_d(s)$. To determine how the closed-loop poles leave p_1 and p_2 and approach the real axis locus, we first compute the angle at which the locus departs from p_1. (Of course, the locus near p_2 will simply be a mirror image about the real axis.) Taking a test point s very near p_1, it is easy to see that the contributions of z_1, z_2, and p_2 to $\angle P(s)F_d(s)$ will remain more or less constant as we move s in a small circle around p_1. Choosing $s = \tilde{p}_1$, a point near p_1, we must have

$$\begin{aligned}
\angle P(\tilde{p}_1)F_d(\tilde{p}_1) &= (2k+1)\pi \qquad k = 0, \pm 1, \pm 2, \ldots \\
&= \angle(\tilde{p}_1 - z_1) + \angle(\tilde{p}_1 - z_2) - \angle(\tilde{p}_1 - p_1) - \angle(\tilde{p}_1 - p_2) \\
&\approx \angle(p_1 - z_1) + \angle(p_1 - z_2) - \theta_d - \angle(p_1 - p_2)
\end{aligned}$$

where θ_d is the departure angle from p_2. Taking $k = 0$, we therefore have

$$\begin{aligned}
\theta_d &= -\pi + \angle(p_1 - z_1) + \angle(p_1 - z_2) - \angle(p_1 - p_2) \\
&= -\pi + \arctan\left(\frac{\sqrt{3}}{0}\right) + \arctan\left(\frac{\sqrt{3}}{1}\right) - \arctan\left(\frac{2\sqrt{3}}{0}\right) \\
&= -\pi + \frac{\pi}{2} + \frac{\pi}{3} - \frac{\pi}{2} \\
&= -\frac{2\pi}{3}.
\end{aligned}$$

By mirror symmetry, the departure angle from p_2 must be $\frac{2\pi}{3}$.

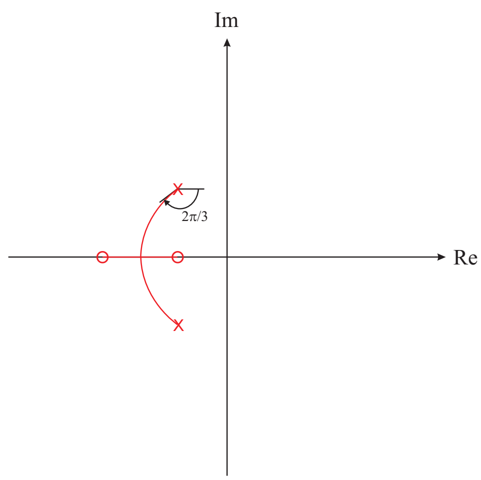

Figure 2: Root locus example #3.

Next, we compute the "breakin point" exactly as we did before, i.e., by using the condition for the existence of a double-pole. From the coalescence condition

$$b'(s)a(s) - a'(s)b(s) = 0,$$

we require that

$$\begin{aligned}
0 &= (2s+3)\left(s^2 + 2s + 4\right) - (2s+2)\left(s^2 + 3s + 2\right) \\
&= -s^2 + 4s + 8.
\end{aligned}$$

Two solutions are $s = 2 \mp 2\sqrt{3}$; only the '$-$' solution lies on the root locus, so the breakin point is $s = 2 - 2\sqrt{3} \approx -1.46$.

- A total of m closed-loop poles approach the zeros of the loop gain $P(s)F_d(s)$ as $K \to \infty$. The remaining $n - m$ closed-loop poles follow asymptotes outward to infinity.

Following is the general procedure for constructing a root locus plot, as adapted from [1].

Step 1. Locate the poles and zeros of the loop gain $P(s)F_d(s)$. First, compute the zeros of the loop gain (the roots of $b(s)$) and place an 'o' at their location in the complex plane. Next, compute the poles of the loop gain (the roots of $a(s)$) and place an 'x' at their location in the complex plane.

Step 2. Determine what, if any, portion of the real axis is part of the root locus. The angle condition requires that the real axis portion of the root locus lies to the left of an odd number of poles and zeros. Equivalently, since complex poles and zeros must occur in conjugate pairs, the real axis portion of the root locus lies to the left of an odd number of *real* poles and zeros.

Step 3. Determine the asymptotes of the root locus. Given that there are m zeros and $n \geq m$ poles, m of the closed-loop poles will approach the loop gain zeros as $K \to \infty$ and the remaining $n - m$ will converge to asymptotes which extend radially to infinity from some starting point on the real axis. The asymptote angles are

$$\frac{(2k+1)\pi}{n-m} \qquad k = 0, \pm 1, \pm 2, \ldots,$$

which can be proved by approximating the loop gain with $\frac{K}{s^{n-m}}$ for large values of $|s|$.

The center of the asymptotes can be computed from a slightly better approximation obtained as follows. Write

$$\begin{aligned} \frac{b(s)}{a(s)} &= \frac{(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_n)} \\ &= \frac{s^m + (-z_1 \cdots - z_m)s^{m-1} + \cdots}{s^n + (-p_1 \cdots - p_n)s^{n-1} + \cdots} \end{aligned}$$

Dividing both the numerator and the denominator by the numerator gives

$$\frac{b(s)}{a(s)} = \frac{1}{s^{n-m} + \left((z_1 + \cdots + z_m) - (p_1 + \cdots + p_n)\right) s^{n-m-1} + \cdots} \tag{1}$$

For large values of $|s|$, this ratio of polynomials can be approximated by

$$\left(s + \frac{(z_1 + \cdots + z_m) - (p_1 + \cdots + p_n)}{n-m}\right)^{-(n-m)}. \tag{2}$$

That is, (1) matches (2) to order s^{n-m-1}. The root locus for *this* (approximate) loop gain consists of $n-m$ rays extending radially from the point

$$\sigma = \frac{(p_1 + \cdots + p_n) - (z_1 + \cdots + z_m)}{n-m}.$$

The real number σ is the center of the asymptotes for root locus corresponding to the true loop gain.

Step 4. Find the breakaway and breakin points. Recall that these points correspond to values of the gain K for which the closed-loop system has multiple closed-loop poles at a particular point. For a double-pole, the condition

$$b'(s)a(s) - a'(s)b(s) = 0$$

must be satisfied. The roots of this algebraic equation give *possible* breakaway or breakin points. To determine whether these are, in fact, breakaway or breakin points, one must check whether these points are actually on the root locus.

Step 5. Determine the angles of departure from the loop gain poles and the angles of arrival at the loop gain zeros. Recall that as $K \to 0$, the root locus approaches the poles of the loop gain and as $K \to \infty$, m branches of the root locus approach the zeros of the loop gain. The angle of departure from the k^{th} loop gain pole p_k can be obtained from the angle condition as

$$\theta_d = \pi + \sum_i \angle(p_k - z_i) - \sum_{j \neq k} \angle(p_k - p_j).$$

That is, the departure angle is π plus the sum of all the angles of vectors pointing from the loop gain zeros to p_k minus the sum of all the angles of vectors pointing from the remaining loop gain poles to p_k.

Similarly, one can use the angle condition to show that the angle of arrival at the k^{th} loop gain zero z_k is

$$\theta_a = \pi - \sum_{i \neq k} \angle(z_k - z_i) + \sum_j \angle(z_k - p_j).$$

□

In general, it is a good idea to also compute the value of K at which the root locus crosses into the right half of the complex plane for the first time. This can be done using the Routh-Hurwitz procedure. The gain value at which the root locus first crosses into the right half plane generally serves as an upper limit on acceptable choices of the parameter K.

To determine the value of K corresponding to a particular closed-loop pole on the root locus, one must use the magnitude condition. Recognizing that

$$\left| K \frac{b(s)}{a(s)} \right| = 1 \qquad \Leftrightarrow \qquad K = \left| \frac{a(s)}{b(s)} \right|,$$

we have, for a particular closed-loop pole $\tilde{s}$,

$$K = \frac{\prod_j |(\tilde{s} - p_j)|}{\prod_i |(\tilde{s} - z_i)|}.$$

References

[1] K. Ogata. *Modern Control Engineering, Fourth Ed.* Prentice Hall, Upper Saddle River, NJ, 2002.