Evans' Rules for Sketching the Root Locus

Absolute and Relative Stability. A control system is called *absolutely stable* if the controlled transfer function $H_d(s)$ from the reference signal $y_d(s)$ to the output signal $y(s)$ has all of its poles in the open left half plane. One technique for determining absolute stability of a control system is the Routh-Hurwitz stability analysis technique. This very useful technique is presented in Section 5-7 of [1].

Absolute stability is an essential quality for a control system, but it says nothing about the performance characteristics of the system, i.e., the transient response. Two "absolutely stable" systems can respond to a step input in very different ways; one might exhibit a very slow, overdamped response while the other exhibits a very fast, underdamped response.

To compare the performance of two absolutely stable systems, it is useful to consider the notion of "relative stability" or "degree of stability." Degree of stability can be rather narrowly defined as the horizontal distance between the imaginary axis and the nearest pole. This distance will typically determine the speed of response of the system, however it tells you nothing more about the nature of that response (e.g., if it the system is overdamped, critically damped, or underdamped). More generally, one may examine the specific locations of the closed-loop poles. Knowing these pole locations gives a good sense of the nature of the system's transient response.

The Root Locus Method. The root locus method, also known as "Evans' rules" in honor of W. R. Evans, is a technique for determining how the poles of a feedback control system move in the complex plane as a parameter is varied. Typically, the parameter is a control gain, although any parameter of interest can be used. (For this reason, the root locus method is useful in dynamical system theory, where one is often interested in sudden changes in a system's qualitative behavior, called "bifurcations," as a parameter varies.)

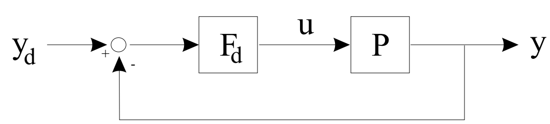

Figure 1: One degree of freedom closed-loop control structure.

Consider the simple feedback control system shown in Figure 1. The closed-loop transfer function is

$$H_d(s) = \frac{y(s)}{y_d(s)} = \frac{P(s)F_d(s)}{1 + P(s)F_d(s)}.$$

Closed-loop poles are values of s for which

$$1 + P(s)F_d(s) = 0.$$

Since $P(s)F_d(s)$ is a function of a complex variable, the equation $P(s)F_d(s) = -1$ can be expressed in terms of the magnitude and phase of $P(s)F_d(s)$:

$$|P(s)F_d(s)| = 1 \qquad \text{and} \qquad \angle P(s)F_d(s) = (2k+1)\pi \quad k = 0, \pm 1, \pm 2, \ldots$$

In words, the magnitude of the "loop gain" is always one and the phase is an odd power of π.

Suppose that $P(s)F_d(s)$ can be written in the form

$$P(s)F_d(s) = K\frac{b(s)}{a(s)}.$$

This would be the case, for example, if $P(s) = b(s)/a(s)$ and $F_d = K$, as for a simple proportional controller. The control structure might be more complicated than this, however we assume that a the multiplicative factor K appears and that this parameter may vary.

The "root locus" is the "locus" of possible roots of the closed-loop transfer function as the multiplicative parameter K is varied. In fact, the entire root locus can be determined from the angle condition alone. The magnitude condition is then used to determine which value of K corresponds to which set of closed-loop poles along the locus of all possible closed-loop poles.

Rather than learn Evans' rules to begin with, it is more illustrative to consider a series of increasingly complicated examples.

Example 1. To begin, we consider the very simple example

$$P(s)F_d(s) = K\frac{1}{s(s+2)}.$$

We will compute the closed-loop poles as explicit functions of K. In general, this is a tedious, and uninformative exercise, but for this simple system it serves to illustrate how closed-loop poles vary as the gain K is varied. The closed-loop transfer function is

$$H_d(s) = \frac{\frac{K}{s(s+2)}}{1+\frac{K}{s(s+2)}}.$$

The closed-loop poles are obtained from

$$\begin{aligned} 0 &= 1 + \frac{K}{s(s+2)} \\ &= s^2 + 2s + K. \end{aligned}$$

They are

$$\begin{aligned} s &= \frac{1}{2}(-2 \pm \sqrt{4-4K}) \\ &= -1 \pm \sqrt{1-K}. \end{aligned}$$

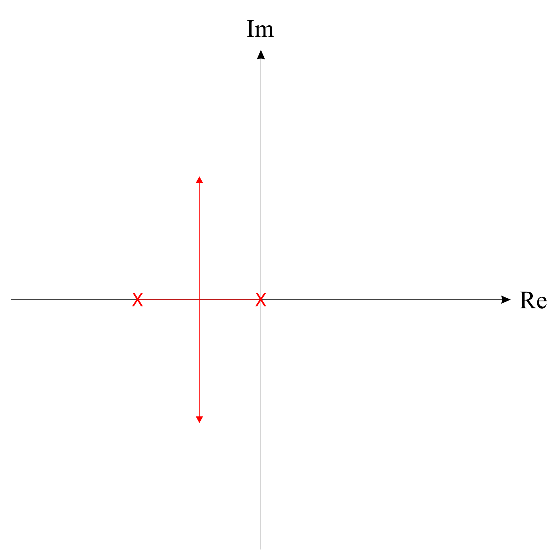

Figure 2: Root locus example #1.

When $0 < K < 1$, there are two distinct poles which are located on the real axis between 0 and -2. When $K = 1$, the poles coalesce at $s = -1$. As K continues to increase, the poles split apart and move in opposite directions parallel to the imaginary axis.

To see that the locus of closed-loop poles shown in Figure 2 can be obtained from the angle condition

$$\angle \frac{1}{s(s+2)} = (2k+1)\pi \quad k = 0, \pm 1, \pm 2, \ldots,$$

we first recall some facts about complex numbers. First, a complex number can be represented in polar form, for example $z = re^{i\theta}$ where r is the radial distance from the origin to the point z and θ is the angle to z measured counter-clockwise from the positive real axis. Consider the complex function

$$C(s) = \frac{(s - z_1)\cdots(s - z_m)}{(s - p_1)\cdots(s - p_n)}.$$

Each term in the numerator can be considered a vector from the zero z_i to the point s. Similarly, each term in the denominator can be considered a vector from the pole p_i to the point s. Each of these vectors has a magnitude and an angle, so we may equivalently write

$$\begin{aligned} C(s) &= \frac{\left(r_{z_1} e^{i\theta_{z_1}}\right)\cdots\left(r_{z_m} e^{i\theta_{z_m}}\right)}{\left(r_{p_1} e^{i\theta_{p_1}}\right)\cdots\left(r_{p_n} e^{i\theta_{p_n}}\right)} \\ &= \left(\frac{r_{z_1}\cdots r_{z_m}}{r_{p_1}\cdots r_{p_n}}\right) e^{i\left(\theta_{z_1}+\cdots+\theta_{z_m}-\theta_{p_1}-\cdots-\theta_{p_n}\right)} \end{aligned}$$

where r_{z_i} (or r_{p_i}) is the magnitude of the vector from z_i (or p_i) to s and θ_{z_i} (or θ_{p_i}) is the angle of the vector from z_i (or p_i) to s.

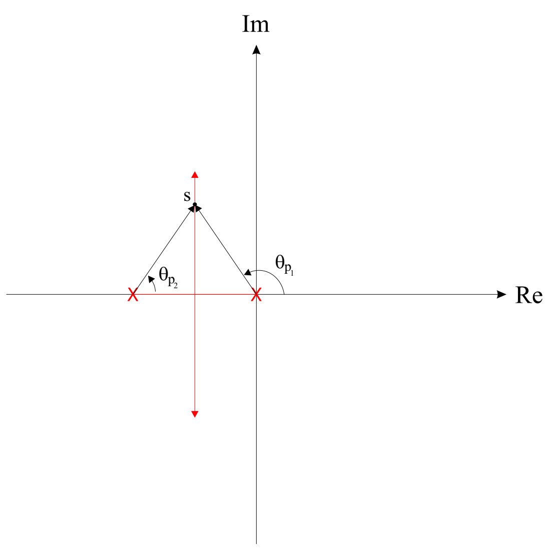

Figure 3: Angle condition for root locus example #1.

Applying these observations to the current example, we find that

$$\angle \frac{1}{s(s+2)} = -\angle s - \angle(s+2). \tag{1}$$

Now, for any point on the real axis to the right of $p_1 = 0$, equation (1) gives zero, which is *not* an odd number times π. Similarly, for any point on the real axis to the left of $p_2 = -2$, equation (1) gives -2π, which is also not an odd number times π. Thus, the real axis to the left of p_2 and to the right of p_1 is not part of the root locus. However, for points between p_2 and p_1, equation (1) gives

$$\angle \frac{1}{s(s+2)} = -\angle s - \angle(s+2) = -\pi - 0,$$

which *is* an odd number times π. Thus, points on the real axis between p_2 and p_1 are part of the root locus.

Considering next the points on the vertical line $s = -1$, we choose a point and determine $\angle \frac{1}{s(s+2)}$. The vectors from p_1 and p_2 to any such point form an isosceles triangle. The sum of the two angles is π for points above the real axis and 3π for points below the real axis, giving $\angle \frac{1}{s(s+2)} = -\pi$ or -3π, respectively. Thus, the line $s = -1$ is part of the root locus.

To find the value of K which corresponds to a particular pair of closed-loop poles, we use the magnitude condition. For example, suppose we would like to choose K so that the closed-loop system has a damping ratio $\zeta = \frac{\sqrt{2}}{2}$. Any pole lying on the radius $\theta = \frac{3\pi}{4}$ in the complex plane has damping ratio $\zeta = \frac{\sqrt{2}}{2}$. Thus, we would like to choose K to give closed-loop poles at

$$s = -1 \pm i \tan\frac{\pi}{4} = -1 \pm i.$$

Choosing a particular pole, say $s = -1 + i$, we substitute into the magnitude condition to obtain

$$\left| \frac{K}{(-1+i)((-1+i)+2)} \right| = 1$$

or

$$K = |(-1+i)(1+i)| = |-2| = 2.$$

Thus, choosing the gain $K = 2$ gives the closed-loop poles $s = -1 \pm i$. $\square$

We have assumed that $P(s)F_d(s)$ can be written in the form

$$P(s)F_d(s) = K\frac{b(s)}{a(s)}.$$

where $b(s)$ has degree m, $a(s)$ has degree $n \geq m$ and where $K > 0$ is a parameter (e.g., a control gain) which may vary.

An important observation is that, as $K \to 0$, the closed-loop poles approach the poles of the loop gain. To see this, write the closed-loop characteristic equation as

$$a(s) + Kb(s) = 0.$$

Clearly, as $K \to 0$ the roots of the polynomial on the left approach the roots of $a(s)$.

One may also observe that, as $K \to \infty$, the closed-loop poles must either diverge to ∞ or approach a zero of the loop gain. To see this, recognize that as $K \to \infty$, $\frac{b(s)}{a(s)}$ must become very small so that the product is -1. There are two ways that $\frac{b(s)}{a(s)}$ can become very small. First, $b(s)$ can go to zero (which happens when s approaches a zero of the loop gain). Second, $a(s)$ can go to infinity (which can only happen when $|s|$ goes to infinity.) In general, m branches of the root locus approach the zeros of the loop gain while the remaining $n - m$ branches go to infinity.

Example 2. Consider the following example from [1]:

$$P(s)F_d(s) = \frac{K}{s(s+1)(s+2)}.$$

This system has poles at $p_1 = 0$, $p_2 = -1$, and $p_3 = -2$. Recalling that

$$\angle P(s)F_d(s) = \sum_{i=1}^{m} \angle(s - z_i) - \sum_{j=1}^{n} \angle(s - p_j),$$

we first consider which, if any, points on the real axis are part of the root locus. For any point to the right of $s = 0$, $\angle P(s)F_d(s) = 0$, so the positive real axis is *not* part of the root locus. For any point $-1 < s < 0$, $\angle P(s)F_d(s) = -\pi$, so these points *are* part of the root locus. For any point $-2 < s < -1$, $\angle P(s)F_d(s) = -2\pi$, so these points are *not* part of the root locus. Finally, for any point $s < -2$, $\angle P(s)F_d(s) = -3\pi$, so these points *are* part of the root locus.

Next, we consider what happens to the root locus as s grows large. In the limit that s grows large, we have

$$\lim_{|s|\to\infty} P(s)F_d(s) = \lim_{|s|\to\infty} \frac{K}{s(s+1)(s+2)} = \lim_{|s|\to\infty} \frac{K}{s^3}.$$

Now, no matter how large $|s|$ is, the angle condition must be satisfied, so we must have

$$\begin{aligned}
\lim_{|s|\to\infty} \angle\, P(s)F_d(s) &= \lim_{r\to\infty} \angle\, P(re^{i\theta})F_d(re^{i\theta}) \\
&\approx \lim_{r\to\infty} \frac{K}{(re^{i\theta})^3} \\
&= \angle\, e^{-i3\theta} \\
&= (2n+1)\pi \qquad n = 0, \pm 1, \pm 2, \ldots
\end{aligned}$$

or

$$\theta = -\frac{2n+1}{3}\pi.$$

Trying $n = 0$ gives $\theta = -\frac{\pi}{3}$. Trying $n = 1$ gives $\theta = -\pi$. Trying $n = 2$ gives $\theta = -\frac{5\pi}{3}$. Other choices of n give repeated angles. In the limit that $|s| \to \infty$, the three closed-loop poles follow asymptotes that extend radially in the directions $\pm\frac{\pi}{3}$ and π.

A logical question is "How large must $|s|$ be before the root locus converges to these asymptotes?" Rephrasing the question, "How does the root locus look for smaller values of $|s|$ given that it converges to these asymptotes as $|s|$ grows large?" A partial answer can be obtained by determining where the three asymptotes are centered. A simple way to determine this is to notice that a slightly more precise approximation for $P(s)F_d(s)$ for large $|s|$ is

$$\lim_{|s|\to\infty} P(s)F_d(s) = \lim_{|s|\to\infty} \frac{K}{(s+1)^3}.$$

To see this, compare the polynomials

$$(s+1)^3 = s^3 + 3s^2 + 3s + 1 \qquad \text{and} \qquad s(s+1)(s+2) = s^3 + 3s^2 + 2s.$$

The two polynomials agree to next-to-highest order. The root locus for the large-$|s|$ approximation is the three asymptotes computed previously, centered at the point $s = -1$. For the true system, the root locus will behave a bit differently for small $|s|$, but we have at least located the origin of the asymptotes which describe the large $|s|$ behavior.

Two of the three asymptotes extend into the right half complex plane, while the third follows the negative real axis. Intuitively, the closed-loop pole which starts (for small K) at $s = -2$ will follow the negative real axis asymptote as K increases. Therefore, the two closed-loop poles which rest on the real axis between $s = -1$ and $s = 0$ must coalesce and split off to follow the asymptotes at $\pm\frac{\pi}{3}$.[1]

[1]They must first coalesce because poles must be either real numbers or complex conjugate pairs and because the closed-loop pole locations vary continuously with K.

To determine precisely where this split occurs, known as a "breakaway point," we recognize that there must be a *double pole* between $s = -1$ and $s = 0$ for some value of K. Recall from System Dynamics that a double-root $s = \tilde{s}$ of a polynomial $C(s)$ satisfies not only $C(\tilde{s}) = 0$, but also $C'(\tilde{s}) = 0$. You can see this by recognizing that $C(s) = (s - \tilde{s})^2 \prod_{i=1}^{n-2} (s - p_i)$ if there is a double pole at $\tilde{s}$.

For the feedback control system, we therefore have not only $a(s) + Kb(s) = 0$, when the closed-loop poles coalesce, but also

$$\frac{d}{ds}\left(a(s) + Kb(s)\right) = 0.$$

Solving for K, the value of the parameter for which the double-pole occurs, we find

$$K = -\frac{a'(s)}{b'(s)}.$$

Substituting back into the condition for a closed-loop pole, we have

$$a(s) - \frac{a'(s)}{b'(s)} b(s) = 0$$

or

$$b'(s)a(s) - a'(s)b(s) = 0 \tag{2}$$

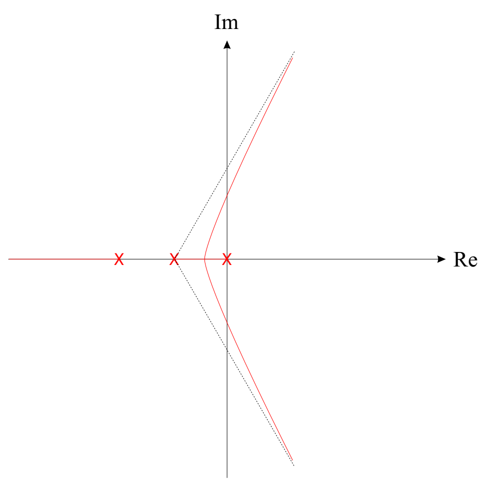

Figure 4: Root locus example #2.

For our system, we have $b(s) = 1$ and $a(s) = s(s + 1)(s + 2)$. Solving (2) for the multiple-pole, we obtain

$$-(3s^2 + 6s + 2) = 0.$$

Two solutions are

$$s = \frac{-6 \pm \sqrt{12}}{6} = -1 \pm \frac{\sqrt{3}}{3}$$

The '$-$' solution is not on the root locus, so the breakaway point must be $s = -1 + \frac{\sqrt{3}}{3} \approx -0.4$.

Note: You can find the value of the gain K at which the root locus passes into the right half plane by performing a Routh-Hurwitz stability analysis and finding conditions on K for stability.

Example 3. Next consider a system for which

$$P(s)F_d(s) = K\frac{s^2 + 3s + 2}{s^2 + 2s + 4}.$$

The loop gain has two zeros at $z_1 = -1$ and $z_2 = -2$ and two poles at $p_{1,2} = -1 \pm i\sqrt{3}$.

First, determine if any closed-loop poles lie on the real axis. The vectors from the two poles to points on the real axis form an isosceles triangle. The sum of their contribution to $\angle PF_d$ is -2π. Similarly, a pair of complex conjugate zeros would contribute 2π to the angle calculation. Thus, we see that *complex conjugate poles and zeros have no impact on whether a given segment of the real axis is part of the root locus.* The real axis poles and zeros alone determine which portion of the real axis is part of the root locus. In fact, one can easily verify that the following statement is true:

- The portion of the real axis which is part of the root locus lies to the left of an odd number of poles and zeros.

For the present example, we observe that the only part of the real axis which contributes to the root locus lies between the zeros at z_1 and z_2. Recall that for small values of K, the closed-loop poles are close to the poles of the loop gain $P(s)F_d(s)$. To determine how the closed-loop poles leave p_1 and p_2 and approach the real axis locus, we first compute the angle at which the locus departs from p_1. (Of course, the locus near p_2 will simply be a mirror image about the real axis.) Taking a test point s very near p_1, it is easy to see that the contributions of z_1, z_2, and p_2 to $\angle P(s)F_d(s)$ will remain more or less constant as we move s in a small circle around p_1. Choosing $s = \tilde{p}_1$, a point near p_1, we must have

$$\begin{aligned} \angle P(\tilde{p}_1)F_d(\tilde{p}_1) &= (2k+1)\pi \qquad k = 0, \pm 1, \pm 2, \ldots \\ &= \angle(\tilde{p}_1 - z_1) + \angle(\tilde{p}_1 - z_2) - \angle(\tilde{p}_1 - p_1) - \angle(\tilde{p}_1 - p_2) \\ &\approx \angle(p_1 - z_1) + \angle(p_1 - z_2) - \theta_d - \angle(p_1 - p_2) \end{aligned}$$

where θ_d is the departure angle from p_2. Taking $k = 0$, we therefore have

$$\begin{aligned} \theta_d &= -\pi + \angle(p_1 - z_1) + \angle(p_1 - z_2) - \angle(p_1 - p_2) \\ &= -\pi + \arctan\left(\frac{\sqrt{3}}{0}\right) + \arctan\left(\frac{\sqrt{3}}{1}\right) - \arctan\left(\frac{2\sqrt{3}}{0}\right) \\ &= -\pi + \frac{\pi}{2} + \frac{\pi}{3} - \frac{\pi}{2} \\ &= -\frac{2\pi}{3}. \end{aligned}$$

By mirror symmetry, the departure angle from p_2 must be $\frac{2\pi}{3}$.

Next, we compute the "breakin point" exactly as we did before, i.e., by using the condition for the existence of a double-pole. From the coalescence condition

$$b'(s)a(s) - a'(s)b(s) = 0,$$

we require that

$$\begin{aligned} 0 &= (2s+3)\left(s^2 + 2s + 4\right) - (2s+2)\left(s^2 + 3s + 2\right) \\ &= -s^2 + 4s + 8. \end{aligned}$$

Two solutions are $s = 2 \mp 2\sqrt{3}$; only the '$-$' solution lies on the root locus, so the breakin point is $s = 2 - 2\sqrt{3} \approx -1.46$.

- A total of m closed-loop poles approach the zeros of the loop gain $P(s)F_d(s)$ as $K \to \infty$. The remaining $n - m$ closed-loop poles follow asymptotes outward to infinity.

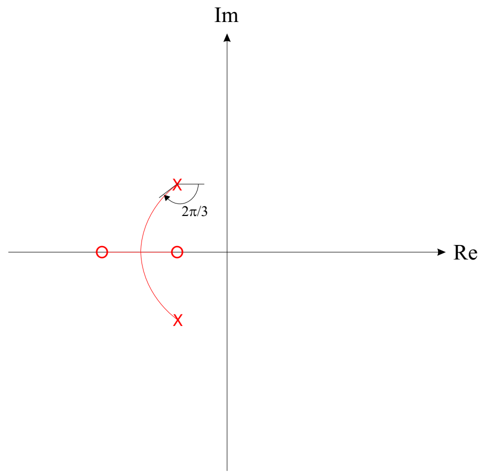

Figure 5: Root locus example #3.

Following is the general procedure for constructing a root locus plot, as adapted from [1].

Step 1. Locate the poles and zeros of the loop gain $P(s)F_d(s)$. First, compute the zeros of the loop gain (the roots of $b(s)$) and place an 'o' at their location in the complex plane. Next, compute the poles of the loop gain (the roots of $a(s)$) and place an 'x' at their location in the complex plane.

Step 2. Determine what, if any, portion of the real axis is part of the root locus. The angle condition requires that the real axis portion of the root locus lies to the left of an odd number of poles and zeros. Equivalently, since complex poles and zeros must occur in conjugate pairs, the real axis portion of the root locus lies to the left of an odd number of *real* poles and zeros.

Step 3. Determine the asymptotes of the root locus. Given that there are m zeros and $n \geq m$ poles, m of the closed-loop poles will approach the loop gain zeros as $K \to \infty$ and the remaining $n - m$ will converge to asymptotes which extend radially to infinity from some starting point on the real axis. The asymptote angles are

$$\frac{(2k+1)\pi}{n-m} \qquad k = 0, \pm 1, \pm 2, \ldots,$$

which can be proved by approximating the loop gain with $\frac{K}{s^{n-m}}$ for large values of $|s|$.

The center of the asymptotes can be computed from a slightly better approximation obtained as follows. Write

$$\begin{aligned} \frac{b(s)}{a(s)} &= \frac{(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_n)} \\ &= \frac{s^m + (-z_1 \cdots - z_m)s^{m-1} + \cdots}{s^n + (-p_1 \cdots - p_n)s^{n-1} + \cdots} \end{aligned}$$

Dividing both the numerator and the denominator by the numerator gives

$$\frac{b(s)}{a(s)} = \frac{1}{s^{n-m} + ((z_1 + \cdots + z_m) - (p_1 + \cdots + p_n))\, s^{n-m-1} + \cdots} \tag{3}$$

For large values of $|s|$, this ratio of polynomials can be approximated by

$$\left(s + \frac{(z_1 + \cdots + z_m) - (p_1 + \cdots + p_n)}{n-m}\right)^{-(n-m)}. \tag{4}$$

That is, (3) matches (4) to order s^{n-m-1}. The root locus for *this* (approximate) loop gain consists of $n-m$ rays extending radially from the point

$$\sigma = \frac{(p_1 + \cdots + p_n) - (z_1 + \cdots + z_m)}{n-m}.$$

The real number σ is the center of the asymptotes for root locus corresponding to the true loop gain.

Step 4. Find the breakaway and breakin points. Recall that these points correspond to values of the gain K for which the closed-loop system has multiple closed-loop poles at a particular point. For a double-pole, the condition

$$b'(s)a(s) - a'(s)b(s) = 0$$

must be satisfied. The roots of this algebraic equation give *possible* breakaway or breakin points. To determine whether these are, in fact, breakaway or breakin points, one must check whether these points are actually on the root locus.

Step 5. Determine the angles of departure from the loop gain poles and the angles of arrival at the loop gain zeros. Recall that as $K \to 0$, the root locus approaches the poles of the loop gain and as $K \to \infty$, m branches of the root locus approach the zeros of the loop gain. The angle of departure from the k^{th} loop gain pole p_k can be obtained from the angle condition as

$$\theta_d = \pi + \sum_i \angle(p_k - z_i) - \sum_{j \neq k} \angle(p_k - p_j).$$

That is, the departure angle is π plus the sum of all the angles of vectors pointing from the loop gain zeros to p_k minus the sum of all the angles of vectors pointing from the remaining loop gain poles to p_k.

Similarly, one can use the angle condition to show that the angle of arrival at the k^{th} loop gain zero z_k is

$$\theta_a = \pi - \sum_{i \neq k} \angle(z_k - z_i) + \sum_j \angle(z_k - p_j).$$

□

In general, it is a good idea to also compute the value of K at which the root locus crosses into the right half of the complex plane for the first time. This can be done using the Routh-Hurwitz procedure. The gain value at which the root locus first crosses into the right half plane generally serves as an upper limit on acceptable choices of the parameter K.

To determine the value of K corresponding to a particular closed-loop pole on the root locus, one must use the magnitude condition. Recognizing that

$$\left| K \frac{b(s)}{a(s)} \right| = 1 \qquad \Leftrightarrow \qquad K = \left| \frac{a(s)}{b(s)} \right|,$$

we have, for a particular closed-loop pole $\tilde{s}$,

$$K = \frac{\prod_j |(\tilde{s} - p_j)|}{\prod_i |(\tilde{s} - z_i)|}.$$

Example: Stabilizing an Inverted Pendulum. The nondimensional equation for an inverted pendulum is

$$\ddot{\theta} - \omega_n^2 \theta = \omega_n^2 u,$$

so the transfer function from torque to angle is

$$G(s) = \frac{\omega_n^2}{s^2 - \omega_n^2}.$$

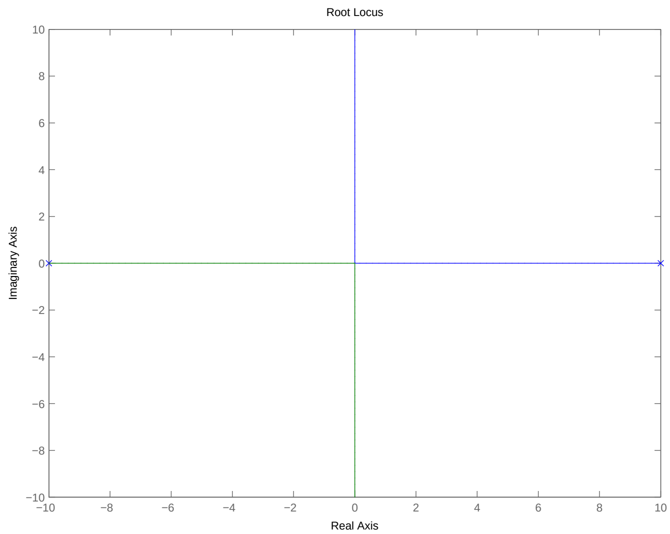

Figure 6: Root locus for proportional feedback.

The system has no zeros and two real-conjugate poles.

We start by applying proportional feedback $F_d(s) = k_p$. The loop gain becomes

$$PF_d = k_p \frac{\omega_n^2}{s^2 - \omega_n^2}.$$

The root locus is shown, for particular parameter values, in Figure 7. Clearly the feedback control law only marginally stabilizes the system, provided k_p is large enough.

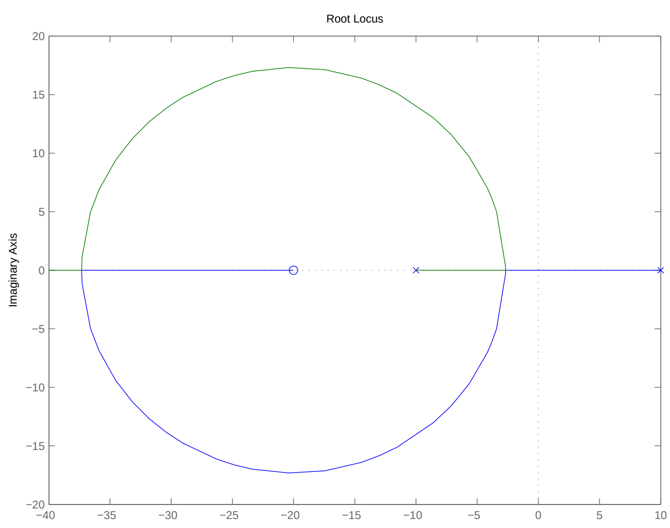

Figure 7: Root locus for proportional-derivative feedback.

Next, we apply proportional-derivative feedback

$$F_d = k_d s + k_p = k_p \left(\frac{k_d}{k_p} s + 1 \right).$$

The loop gain becomes

$$PF_d = -k_p \frac{\left(\frac{1}{\mu} s + 1\right) \omega_n^2}{s^2 - \omega_n^2}$$

where $\mu = \frac{k_p}{k_d}$. The compensator introduces a new loop-gain zero at $-\mu$. We will assume that μ remains constant as k_p is varied. (That is, k_d varies in direct proportion to k_p.) The root locus is shown in Figure 7. Clearly, the feedback controller stabilizes the system. The independent freedom in k_p and μ allows one to obtain any desired closed-loop pole locations (provided the poles are both real or are complex conjugates).

Example: Longitudinal Autopilot. The longitudinal dynamics of the airplane shown in Figure 8 are described by the following equations

$$\begin{pmatrix} \begin{pmatrix} \dot{V} \\ \dot{\alpha} \end{pmatrix} \\ \dot{\theta} \\ \dot{q} \end{pmatrix} = \left\{ \begin{matrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{V} \end{pmatrix} \left(\begin{pmatrix} \mathcal{D} \\ \mathcal{L} \end{pmatrix} + \boldsymbol{R}^T(\alpha) \begin{pmatrix} u_1 \\ \epsilon u_2 \end{pmatrix} + \boldsymbol{R}^T(\alpha)\boldsymbol{R}(\theta) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \\ q \\ u_2 \end{matrix} \right\},$$

where $\boldsymbol{R}(\sigma)$ is the proper rotation matrix

$$\boldsymbol{R}(\sigma) = \begin{pmatrix} \cos\sigma & -\sin\sigma \\ \sin\sigma & \cos\sigma \end{pmatrix}$$

and the aerodynamic forces satisfy

$$\mathcal{D} = -K_D V^2 \alpha^2 \tag{5}$$
$$\mathcal{L} = -K_L V^2 (1 + \alpha). \tag{6}$$

(Note: These equations have been normalized. All quantities are dimensionless.) The inputs are u_1, representing thrust, and u_2, representing the elevator deflection. The small positive scalar ϵ accounts for a slight downward force due to positive deflections of the elevator.

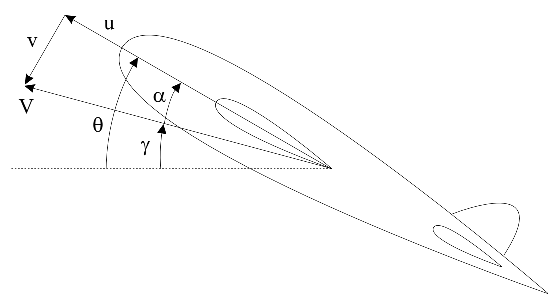

Figure 8: Longitudinal aircraft motion.

Define the state vector $\boldsymbol{x} = [V,\ \alpha,\ \theta,\ q]^T$. Assuming that $K_L = 1/\tilde{V}^2$, we linearize the dynamics about the equilibrium $\boldsymbol{x} = [\tilde{V},\ 0,\ 0,\ 0]^T$. A somewhat tedious series of calculations gives the linearized dynamics

$$\frac{d}{dt}\boldsymbol{\Delta x} = \begin{pmatrix} 0 & 1 & -1 & 0 \\ -\frac{2}{\tilde{V}^2} & -\frac{1}{\tilde{V}} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \boldsymbol{\Delta x} + \begin{pmatrix} 1 & 0 \\ 0 & \frac{\epsilon}{\tilde{V}} \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \boldsymbol{\Delta u}.$$

Suppose we take the output of interest to be flight path angle γ:

$$y = \gamma = [0,\ -1,\ 1,\ 0]\boldsymbol{x}.$$

The transfer function from the elevator u_2 to the flight path angle γ is

$$G(s) = [0,\ -1,\ 1,\ 0] \begin{pmatrix} s & -1 & 1 & 0 \\ \frac{2}{\tilde{V}^2} & s + \frac{1}{\tilde{V}} & 0 & 0 \\ 0 & 0 & s & -1 \\ 0 & 0 & 0 & s \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \frac{\epsilon}{\tilde{V}} \\ 0 \\ 1 \end{pmatrix} = \frac{s\left(-\frac{\epsilon}{\tilde{V}}s^2 + s + \frac{1}{\tilde{V}}\right)}{s^2\left(s^2 + \frac{1}{\tilde{V}}s + \frac{2}{\tilde{V}^2}\right)}$$

$$= \quad -\frac{\left(\frac{\epsilon}{\tilde{V}}s^2 - s - \frac{1}{\tilde{V}}\right)}{s\left(s^2 + \frac{1}{\tilde{V}}s + \frac{2}{\tilde{V}^2}\right)}.$$

Note that this transfer function is *nonminimum phase* for all $\epsilon > 0$. Recall that a stable, non-minimum phase system initially responds to a positive step input in the negative direction. Physically, the downward force on the elevator and tail fin due to a pitch-up command cause the airplane, initially, to accelerate downward. This results in a negative flight path angle until the integrated effect of the tail moment increases the aircraft pitch angle sufficiently to provide upward lift.

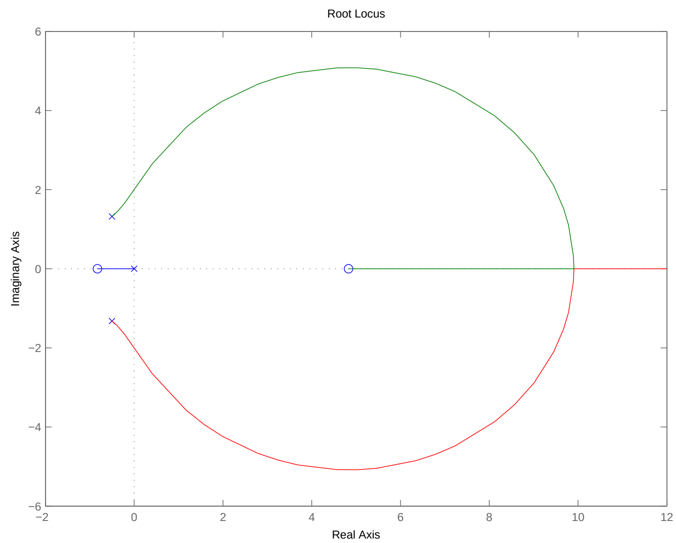

Figure 9: Root locus for proportional feedback.

We would like to design an "autopilot" to regulate the flight path angle for an aircraft with flight conditions and aircraft parameter values given by

$$\tilde{V} = 1, \qquad \text{and} \qquad \epsilon = 0.25.$$

(In fact, the aircraft is already stable, but its motion is very lightly damped. We start by applying proportional feedback $F_d(s) = k_p$. The loop gain becomes

$$PF_d = -k_p \frac{\left(\frac{\epsilon}{\tilde{V}}s^2 - s - \frac{1}{\tilde{V}}\right)}{s\left(s^2 + \frac{1}{\tilde{V}}s + \frac{2}{\tilde{V}^2}\right)}$$

Consider the root locus shown in Figure 9. Observe that straight proportional feedback drives this slightly stable system unstable! The right half plane zero draws the branches of the locus into the right half plane.

Notice that the root locus in Figure 9 appears not to satisfy Evan's rules. For example, the real axis locus does *not* lie to the left of an odd number of poles and zeros. This is a consequence of the negative sign in the numerator, the very term which makes the system nonminimum phase. Essentially, it is as if we have changed the sign of the proportional gain and are now applying *positive* feedback. For positive feedback, the angle condition changes to

$$\angle PF_d = 2k\pi \qquad k = 0, \pm 1, \pm 2, \ldots,$$

and Evan's rules change accordingly.

To stabilize the system, suppose we introduce an additional left half plane zero in the loop gain by applying proportional derivative feedback. This additional zero will have the effect of drawing the branches of the

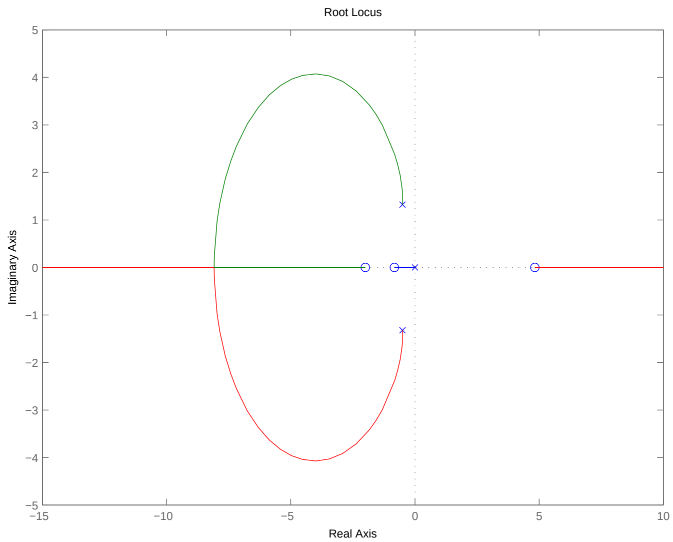

Figure 10: Root locus for proportional-derivative feedback.

root locus into the left half plane. We choose

$$F_d = k_d s + k_p = k_p \left(\frac{k_d}{k_p} s + 1 \right).$$

The loop gain becomes

$$PF_d = -k_p \frac{\left(\frac{1}{\mu}s + 1\right)\left(\frac{\epsilon}{\tilde{V}}s^2 - s - \frac{1}{\tilde{V}}\right)}{s\left(s^2 + \frac{1}{\tilde{V}}s + \frac{2}{\tilde{V}^2}\right)}.$$

where, once again, $\mu = \frac{k_p}{k_d}$.

Figure 10 shows the root locus for this system. The branch coming into the RHP zero from the right is an extension of the branch moving the left. It is merely an artifact. (Root loci for nonminimum phase systems are just *weird*.) Thus, we see that proportional derivative control provides stable regulation of the flight path angle.

Example: Parametric Stability Analysis for Watt's Regulator. Watt's regulator is an early example of mechanical feedback control. This apparatus can be simply modeled as a planar pendulum which is made to rotate about its vertical axis by some device, such as a steam engine, whose speed is to be controlled. The speed of the engine is governed by the regulator through a mechanical linkage between the pendulum and the engine throttle. A simple dynamic model is

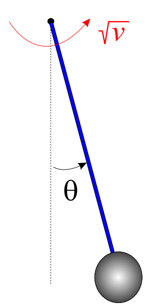

Figure 11: A simple model for Watt's regulator.

$$\ddot{\theta} + b\dot{\theta} + \sin\theta\left(\omega_c^2 - v\cos\theta\right) = 0 \tag{7}$$

$$\dot{v} + av = a\left(\tilde{v} - k(\theta - \tilde{\theta})\right), \tag{8}$$

where θ is the elevation angle between the pendulum and the vertical axis and v represents the (square of the) angular speed of the regulator shaft. The nominal value of v is $\tilde{v}$ and the nominal value of θ is

$$\tilde{\theta} = \arccos\left(\frac{\omega_c^2}{\tilde{v}}\right).$$

The parameter $a > 0$ determines the response time of the engine. The constant $b > 0$ is a damping parameter for the regulator and $\omega_c > 0$ is the undamped natural frequency of planar swinging motion (when $v \equiv 0$). The parameter k is a control gain.

As manufacturing processes improved in the latter half of the nineteenth century, friction in mechanical linkages was dramatically reduced. Coincidentally, new machines whose speed was controlled by Watt regulators began to exhibit undesired speed oscillations due to oscillations in the elevation angle θ. This behavior was referred to as "hunting." Let's use a root locus plot to investigate this phenomenon. To do so, suppose that, for a particular regulator,

$$a = \frac{61}{81} \qquad \text{and} \qquad k = 80\sqrt{3}$$

and that the regulator's nominal (equilibrium) state is

$$(\theta, \dot{\theta}, v)_e = (\tilde{\theta}, 0, \tilde{v}) = \left(\frac{\pi}{3}, 0, 80\right).$$

Linearizing the dynamics about this equilibrium and computing the characteristic equation for the state matrix gives

$$\begin{aligned} 0 &= 81\lambda^3 + (61 + 81b)\lambda^2 + (4860 + 61b)\lambda + 4880 \\ &= \left((\lambda+1)(81\lambda^2 - 20\lambda + 4880)\right) + b(81\lambda^2 + 61\lambda) \end{aligned}$$

or

$$0 = 1 + b\,\frac{81\lambda^2 + 61\lambda}{(\lambda+1)(81\lambda^2 - 20\lambda + 4880)}.$$

The key observation is that the eigenvalue equation can be written in a way which resembles the characteristic equation for a feedback control system with loop gain

$$\frac{81\lambda^2 + 61\lambda}{(\lambda+1)(81\lambda^2 - 20\lambda + 4880)}.$$

Evans' rules apply equally well to this problem. Note that the "loop gain" has two zeros at $\lambda = 0$ and at $\lambda = -\frac{61}{81}$. There are also three loop gain poles: at $s = -1$ and at

$$\begin{aligned} s &= \frac{20 \pm \sqrt{400 - 81 \cdot 4880}}{162} \\ &\approx 0.123 \pm i3.879. \end{aligned}$$

Note that the "closed-loop poles," that is the characteristic values, have positive real part for small values of b. As b is increased, two branches of the root locus approach the two loop gain zeros and the third follows the asymptote along the negative real axis. The portion of the root locus which lies on the real axis lies between the zeros at $s = 0$ and $s = -\frac{61}{81}$ and to the left of the pole at $s = -1$.

We know that the three branches of the root locus leave the poles of the loop gain and that two of these branches somehow approach the two zeros while the third diverges to infinity along the negative real axis.

One way this could happen is that the two branches originating at the complex conjugate poles could coalesce between $s = 0$ and $s = -\frac{61}{81}$. However, there is another possibility which becomes apparent when we attempt to compute the break-in point. To compute the break-in point, we define

$$\begin{aligned} b(s) &= 81s^2 + 61s \\ a(s) &= 81s^3 + 61s^2 + 4860s + 4880 \end{aligned}$$

and compute

$$\begin{aligned} b'(s)a(s) - a'(s)b(s) &= (162s + 61)\left(81s^3 + 61s^2 + 4860s + 4880\right) - \left(243s^2 + 122s + 4860\right)\left(81s^2 + 61s\right) \\ &= -6561s^4 - 9882s^3 + 389939s^2 + 790560s + 297680 \end{aligned}$$

which has roots at

$$s = -7.432, \qquad s = -1.529, \qquad s = -0.502, \qquad \text{and} \qquad s = 7.957.$$

Note that *all but the last point lie on the root locus.* The root locus is clearly more complicated than we first imagined. A simple exercise in logic shows that the only feasible shape for the root locus is the one depicted in Figure 12. Because branches of the root locus leave loop gain poles as b increases, the point $s = -1.529$ must be a break-away point rather than a break-in point. The remaining two points are obviously break-in points. Suppose the two branches which leave the complex conjugate poles were to break in at $s = -0.502$ and converge to the zeros, as we originally hypothesized. Then we would have a contradiction between the fact that the real axis to the left of $s = -1$ is part of the root locus and the fact that there is a break-away and break-in point to the left of the pole at $s = -1$. (Note: a single pole cannot suddenly split and become two poles!)

The angle of departure for the pole at $0.123 + i3.879$ is

$$\begin{aligned} \theta_d &= \pi + 1.539 + 1.349 - 1.289 - \frac{\pi}{2} \\ &= 3.170 \end{aligned}$$

and, by mirror symmetry, the angle of departure for its complex conjugate is -3.170 radians.

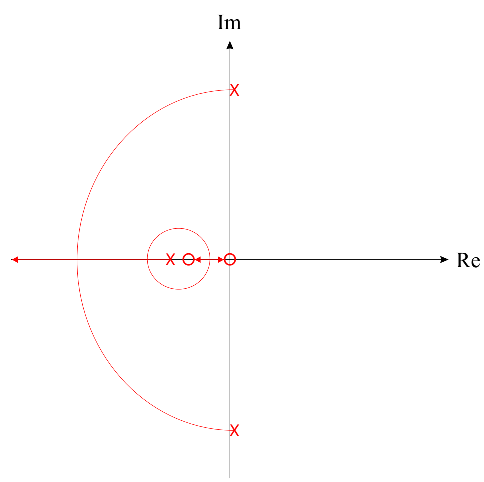

Figure 12: Root locus as b varies from 0 to ∞. Note that as $b \to 0$, the closed loop eigenvalues move into the right half plane.

The hunting phenomenon is a consequence of the fact that, while the damping coefficient had previously been sufficiently large to ensure stability of the closed-loop system, improvements in machining lowered b

below a critical value for stability. The oscillatory behavior referred to as “hunting” is a consequence of what dynamical systems theorists call a “Hopf bifurcation” in which a complex conjugate pair of eigenvalues crosses over the imaginary axis as a “bifurcation parameter” is varied. The equilibrium about which the dynamics are linearized becomes unstable and the nonlinear system begins to exhibit a periodic oscillation about the equilibrium state. Note that this oscillation is not predicted by the linearized dynamics; the right half plane poles of the linearized system would suggest that the state diverges exponentially, but this does not happen in reality. The oscillation is a nonlinear phenomenon referred to as a “limit cycle.”

References

[1] K. Ogata. *Modern Control Engineering, Fourth Ed.* Prentice Hall, Upper Saddle River, NJ, 2002.