Linearized Longitudinal Dynamics

$$\dot{\boldsymbol{x}}_\mathrm{L} = \boldsymbol{A}_\mathrm{L}\boldsymbol{x}_\mathrm{L} + \boldsymbol{B}_\mathrm{L}\boldsymbol{u}_\mathrm{L} \quad \text{where} \quad \boldsymbol{x}_\mathrm{L} = \begin{pmatrix} \Delta x \\ \Delta z \\ \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{pmatrix} \quad \text{and} \quad \boldsymbol{u}_\mathrm{L} = \begin{pmatrix} \Delta\delta e \\ \Delta\delta T \end{pmatrix} \tag{1}$$

and where the state and input matrices are

$$\boldsymbol{A}_\mathrm{L} = \left(\begin{array}{cc|cccc} 0 & 0 & \cos\theta_0 & \sin\theta_0 & 0 & -u_0\sin\theta_0 \\ 0 & 0 & -\sin\theta_0 & \cos\theta_0 & 0 & -u_0\cos\theta_0 \\ \hline 0 & 0 & \frac{1}{m}X_u & \frac{1}{m}X_w & 0 & -g\cos\theta_0 \\ 0 & 0 & \frac{Z_u}{m-Z_{\dot{w}}} & \frac{Z_w}{m-Z_{\dot{w}}} & \frac{(Z_q+mu_0)}{m-Z_{\dot{w}}} & -\frac{mg\sin\theta_0}{m-Z_{\dot{w}}} \\ 0 & 0 & \frac{1}{I_\mathrm{y}}\left(M_u + \frac{M_{\dot{w}}Z_u}{m-Z_{\dot{w}}}\right) & \frac{1}{I_\mathrm{y}}\left(M_w + \frac{M_{\dot{w}}Z_w}{m-Z_{\dot{w}}}\right) & \frac{1}{I_\mathrm{y}}\left(M_q + \frac{M_{\dot{w}}(Z_q+mu_0)}{m-Z_{\dot{w}}}\right) & \frac{1}{I_\mathrm{y}}\left(\frac{M_{\dot{w}}(-mg\sin\theta_0)}{m-Z_{\dot{w}}}\right) \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right) \tag{2}$$

and

$$\boldsymbol{B}_\mathrm{L} = \left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hline \frac{1}{m}X_{\delta e} & \frac{1}{m}X_{\delta T} \\ \frac{Z_{\delta e}}{m-Z_{\dot{w}}} & \frac{Z_{\delta T}}{m-Z_{\dot{w}}} \\ \frac{1}{I_\mathrm{y}}\left(M_{\delta e} + \frac{M_{\dot{w}}Z_{\delta e}}{m-Z_{\dot{w}}}\right) & \frac{1}{I_\mathrm{y}}\left(M_{\delta T} + \frac{M_{\dot{w}}Z_{\delta T}}{m-Z_{\dot{w}}}\right) \\ 0 & 0 \end{array}\right) \tag{3}$$

Since dynamic stability does not depend on x and z, and since the final four equations in (1) are independent of these state components, we may consider the reduced system

$$\dot{\bar{\boldsymbol{x}}}_\mathrm{L} = \bar{\boldsymbol{A}}_\mathrm{L}\bar{\boldsymbol{x}}_\mathrm{L} + \bar{\boldsymbol{B}}_\mathrm{L}\bar{\boldsymbol{u}}_\mathrm{L} \quad \text{where} \quad \bar{\boldsymbol{x}}_\mathrm{L} = \begin{pmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{pmatrix} \quad \text{and} \quad \bar{\boldsymbol{u}}_\mathrm{L} = \begin{pmatrix} \Delta\delta e \\ \Delta\delta T \end{pmatrix}.$$

The state matrix $\bar{\boldsymbol{A}}_\mathrm{L}$ is the lower right block of (2) and the input matrix $\bar{\boldsymbol{B}}_\mathrm{L}$ is the lower block of (3). The dimensional stability derivatives appearing in these expressions may be computed as shown in Table 1.

Table 1: Longitudinal Dimensional Stability Derivatives

	$X_{(\cdot)}$	$Z_{(\cdot)}$	$M_{(\cdot)}$
u	$\frac{1}{2}\rho u_0 S\left[2\left(-C_{D_0} + C_{T_0}\right) + \left(-C_{D_u} + C_{T_u}\right)\right]$	$-\frac{1}{2}\rho u_0 S\left(2C_{L_0} + C_{L_u}\right)$	$\frac{1}{2}\rho u_0 S\bar{c}C_{m_u}$
w	$\frac{1}{2}\rho u_0 S\left(-C_{D_\alpha} + C_{L_0}\right)$	$-\frac{1}{2}\rho u_0 S\left(C_{D_0} + C_{L_\alpha}\right)$	$\frac{1}{2}\rho u_0 S\bar{c}C_{m_\alpha}$
α	u_0X_w	u_0Z_w	u_0M_w
q	≈ 0	$-\frac{1}{4}\rho u_0 S\bar{c}C_{L_q}$	$\frac{1}{4}\rho u_0 S\bar{c}^2C_{m_q}$
$\dot{w}$	≈ 0	$-\frac{1}{4}\rho S\bar{c}C_{L_{\dot{\alpha}}}$	$\frac{1}{4}\rho S\bar{c}^2C_{m_{\dot{\alpha}}}$
$\dot{\alpha}$	≈ 0	$u_0Z_{\dot{w}}$	$u_0M_{\dot{w}}$
δe	≈ 0	$\frac{1}{2}\rho u_0^2 S\left(-C_{L_{\delta e}}\right)$	$\frac{1}{2}\rho u_0^2 S\bar{c}C_{m_{\delta e}}$

Linearized Lateral-Directional Dynamics

$$\dot{\boldsymbol{x}}_{\mathrm{LD}} = \boldsymbol{A}_{\mathrm{LD}}\boldsymbol{x}_{\mathrm{LD}} + \boldsymbol{B}_{\mathrm{LD}}\boldsymbol{u}_{\mathrm{LD}} \quad \text{where} \quad \boldsymbol{x}_{\mathrm{LD}} = \begin{pmatrix} \Delta y \\ \Delta\psi \\ \Delta v \\ \Delta p \\ \Delta r \\ \Delta\phi \end{pmatrix} \quad \text{and} \quad \boldsymbol{u}_{\mathrm{LD}} = \begin{pmatrix} \Delta\delta a \\ \Delta\delta r \end{pmatrix}, \tag{4}$$

and where the state and input matrices are

$$\boldsymbol{A}_{\mathrm{LD}} = \left(\begin{array}{cc:cccc} 0 & u_0\cos\theta_0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sec\theta_0 & 0 \\ \hdashline 0 & 0 & \frac{1}{m}Y_v & \frac{1}{m}Y_p & \frac{1}{m}Y_r - u_0 & g\cos\theta_0 \\ 0 & 0 & \frac{1}{I_x}\mathcal{L}_v & \frac{1}{I_x}\mathcal{L}_p & \frac{1}{I_x}\mathcal{L}_r & 0 \\ 0 & 0 & \frac{1}{I_z}\mathcal{N}_v & \frac{1}{I_z}\mathcal{N}_p & \frac{1}{I_z}\mathcal{N}_r & 0 \\ 0 & 0 & 0 & 1 & \tan\theta_0 & 0 \end{array}\right) \quad \text{and} \quad \boldsymbol{B}_{\mathrm{LD}} = \left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hdashline \frac{1}{m}Y_{\delta a} & \frac{1}{m}Y_{\delta r} \\ \frac{1}{I_x}\mathcal{L}_{\delta a} & \frac{1}{I_x}\mathcal{L}_{\delta r} \\ \frac{1}{I_z}\mathcal{N}_{\delta a} & \frac{1}{I_z}\mathcal{N}_{\delta r} \\ 0 & 0 \end{array}\right). \tag{5}$$

In the matrices above,

$$\begin{aligned} \mathcal{L}_{(\cdot)} &= \frac{I_x}{I_{\mathrm{x}}I_{\mathrm{z}} - I_{\mathrm{xz}}^2}\left(I_{\mathrm{z}}L_{(\cdot)} + I_{\mathrm{xz}}N_{(\cdot)}\right) \\ \mathcal{N}_{(\cdot)} &= \frac{I_z}{I_{\mathrm{x}}I_{\mathrm{z}} - I_{\mathrm{xz}}^2}\left(I_{\mathrm{xz}}L_{(\cdot)} + I_{\mathrm{x}}N_{(\cdot)}\right). \end{aligned}$$

Since lateral-directional dynamic stability does not depend on y and ψ, and since the final four equations in (4) are independent of these state components, we may consider the reduced system

$$\dot{\bar{\boldsymbol{x}}}_{\mathrm{LD}} = \bar{\boldsymbol{A}}_{\mathrm{LD}}\bar{\boldsymbol{x}}_{\mathrm{LD}} + \bar{\boldsymbol{B}}_{\mathrm{LD}}\bar{\boldsymbol{u}}_{\mathrm{LD}} \quad \text{where} \quad \bar{\boldsymbol{x}}_{\mathrm{LD}} = \begin{pmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta\phi \end{pmatrix} \quad \text{and} \quad \bar{\boldsymbol{u}}_{\mathrm{LD}} = \begin{pmatrix} \Delta\delta a \\ \Delta\delta r \end{pmatrix}.$$

The state matrix $\bar{\boldsymbol{A}}_{\mathrm{LD}}$ is the lower right block of $\boldsymbol{A}_{\mathrm{LD}}$ in (5) and the input matrix $\bar{\boldsymbol{B}}_{\mathrm{LD}}$ is the lower block of $\boldsymbol{B}_{\mathrm{LD}}$ in (5). The dimensional stability derivatives appearing in these expressions may be computed as shown in Table 2.

Table 2: Lateral-Directional Dimensional Stability Derivatives

	$Y_{(\cdot)}$	$L_{(\cdot)}$	$N_{(\cdot)}$
v	$\frac{1}{2}\rho u_0 S C_{Y_\beta}$	$\frac{1}{2}\rho u_0 b S C_{l_\beta}$	$\frac{1}{2}\rho u_0 b S C_{n_\beta}$
β	$u_0 Y_v$	$u_0 L_v$	$u_0 N_v$
p	$\frac{1}{4}\rho u_0 b S C_{Y_p}$	$\frac{1}{4}\rho u_0 b^2 S C_{l_p}$	$\frac{1}{4}\rho u_0 b^2 S C_{n_p}$
r	$\frac{1}{4}\rho u_0 b S C_{Y_r}$	$\frac{1}{4}\rho u_0 b^2 S C_{l_r}$	$\frac{1}{4}\rho u_0 b^2 S C_{n_r}$
δa	$\frac{1}{2}\rho u_0^2 S C_{Y_{\delta a}}$	$\frac{1}{2}\rho u_0^2 b S C_{l_{\delta a}}$	$\frac{1}{2}\rho u_0^2 b S C_{n_{\delta a}}$
δr	$\frac{1}{2}\rho u_0^2 S C_{Y_{\delta r}}$	$\frac{1}{2}\rho u_0^2 b S C_{l_{\delta r}}$	$\frac{1}{2}\rho u_0^2 b S C_{n_{\delta r}}$