Review of Linear, Time-Invariant Ordinary Differential Equations

First order, homogeneous, LTI ODEs. Following is the normal form for a homogeneous, linear, time-invariant ordinary differential equation of first order.

$$\dot{x} + ax = 0, \qquad x(t_0) = x_0. \tag{1}$$

The dot indicates differentiation with respect to time. Suppose we wish to solve (1) for $x(t)$.

Approach #1: Use an *integrating factor*. That is, multiply by a factor that will make the left hand side an exact differential.

$$\begin{aligned} e^{at}(\dot{x} + ax) &= 0 \\ \dot{x}e^{at} + axe^{at} &= 0 \\ \frac{d}{dt}\left(xe^{at}\right) &= 0 \end{aligned}$$

Integrating both sides from t_0 to t gives

$$\int_{t_0}^{t} \frac{d}{d\tau}\left(x(\tau)e^{a\tau}\right) d\tau = \int_{t_0}^{t} 0 \, d\tau \qquad \Rightarrow \qquad x(t)e^{at} - x(t_0)e^{at_0} = 0 \qquad \Rightarrow \qquad x(t) = x_0 e^{-a(t-t_0)}$$

Comments:

- The integrating factor also works when there is forcing on the right hand side and when the equation is linear time varying (i.e., when a depends on t).
- As stated, the approach only works for first order linear equations. However, it can be generalized to *systems* of first order linear equations. (Recall that any number of linear equations of any order can be rewritten as a set of first order linear equations.)

Approach #2: Motivated by divine insight, *assume* a solution of the form

$$x(t) = ce^{\lambda t}. \tag{2}$$

(Actually, this form seems quite reasonable given the solution in Approach #1.) Substituting into (1) gives

$$\begin{aligned} \dot{x} + ax &= 0 \\ \frac{d}{dt}\left(ce^{\lambda t}\right) + a\left(ce^{\lambda t}\right) &= 0 \\ (\lambda + a)ce^{\lambda t} &= 0 \\ (\lambda + a)x(t) &= 0. \end{aligned}$$

Now, $x(t)$ will not be zero, in general; only in the special case that $x_0 = 0$ will $x(t)$ be zero. Since the identity above must hold for *any* initial state, it follows that

$$\lambda = -a.$$

To determine c, we evaluate $x(t)$ given in (2) at t_0 and use the initial condition:

$$x(t_0) = ce^{-at_0} \qquad \Rightarrow \qquad c = x_0 e^{at_0}.$$

Substituting the values of λ and c into (2) gives the solution

$$x(t) = x_0 e^{-a(t-t_0)}.$$

Approach #3: Since this is a linear, time-invariant system we may use the Laplace transform to convert the differential equation in time into an algebraic equation in s. Having expanded the solution $X(s)$ in a series of partial fractions, we may then transform back to the time-domain. Assume that $t_0 = 0$. There is no loss of generality in doing this; because the system is time-invariant we simply change the time coordinate, shifting the origin of time such that $t_0 = 0$. Taking the Laplace transform of the equation gives

$$\mathcal{L}\{\dot{x} + ax = 0\} \qquad \Rightarrow \qquad (sX(s) - x_0) + aX(s) = 0 \qquad \Rightarrow \qquad X(s) = \frac{x_0}{s+a}.$$

Taking the inverse Laplace transform gives

$$x(t) = \mathcal{L}^{-1}\left\{\frac{x_0}{s+a}\right\} = x_0 e^{-at}.$$

To better understand the nature of the solution, define the *time constant*

$$T = \frac{1}{a}.$$

Dividing through by the initial state (which we assume is nonzero), we have

$$\frac{x(t)}{x_0} = e^{-\frac{t}{T}}.$$

If $a > 0$ (so that $T > 0$), then the solution decays with time. In this case, at time $t = T$ (that is, after one time constant has elapsed), $x(t)$ is only 37% of its initial value. Slightly before this time, $x(t)$ is exactly one-half its initial value. To determine this *time to half-amplitude* t_{half}, observe that

$$\frac{x(t_{\text{half}})}{x_0} = \frac{1}{2} = e^{-\frac{t_{\text{half}}}{T}} \qquad \Rightarrow \qquad t_{\text{half}} = -\ln\left(\frac{1}{2}\right)T \approx 0.69T.$$

Suppose that $a < 0$, so that $T < 0$. Then $x(t)$ grows without bound. The time required for $x(t)$ to grow to twice its initial amplitude is obtained as follows:

$$\frac{x(t_{\text{double}})}{x_0} = 2 = e^{-\frac{t_{\text{double}}}{T}} \qquad \Rightarrow \qquad t_{\text{double}} = -\ln(2)\,T \approx 0.69|T|.$$

Now suppose $x_0 = 0$. Then $x(t) = 0$ for all time, regardless of the value of a. Thus, $x(t) = 0$ is an *equilibrium*. *Stability* of this equilibrium depends on the sign of the constant a. If $a > 0$, then trajectories starting near the equilibrium approach it asymptotically and we say that the equilibrium is stable. If $a < 0$, then trajectories starting near the equilibrium diverge from it and we say that the equilibrium is unstable.

First order, heterogeneous, LTI ODEs. Following is the normal form for a heterogeneous, linear, constant coefficient ordinary differential equation of first order.

$$\dot{x} + ax = f(t), \qquad x(t_0) = x_0. \tag{3}$$

Suppose we wish to solve (1) for $x(t)$. (Assume that $a > 0$.)

Approach #1: Use an integrating factor.

$$e^{at}(\dot{x} + ax) = e^{at} f(t) \qquad \Rightarrow \qquad \frac{d}{dt}\left(xe^{at}\right) = e^{at} f(t).$$

Integrating both sides from t_0 to t gives

$$x(t)e^{at} - x(t_0)e^{at_0} = \int_{t_0}^{t} e^{a\tau} f(\tau) d\tau \qquad \Rightarrow \qquad x(t) = x_0 e^{-a(t-t_0)} + \int_{t_0}^{t} e^{-a(t-\tau)} f(\tau) d\tau \tag{4}$$

Note that $x(t)$ is the *sum* of a term involving the initial condition and a term involving the force $f(t)$. This observation reflects the *principle of superposition* for solutions to linear equations. This principle is useful in solving more general problems (i.e., heterogeneous equations of arbitrary order).

Approach #2: The *method of undetermined coefficients* applies to forced LTI equations where $f(t)$ solves *some* linear differential equation. Equivalently, the method applies to forced LTI equations where a *finite set S contains $f(t)$ and its derivatives of all orders.* For example, the forcing function

$$f(t) = t\sin(\omega t)$$

is admissible because the finite set

$$S = \{t\sin(\omega t),\ t\cos(\omega t),\ \sin(\omega t),\ \cos(\omega t)\}$$

contains $f(t)$ and its derivatives of all orders. (Or rather, $f(t)$ and its derivatives of all orders may be constructed by linear combinations of the elements of S.) Note that each element of S is *linearly independent*; no element can be formed by a linear combination of the remaining elements. (You may check this by computing the "Wronskian determinant" and making sure it is not zero for all t; see your textbook on differential equations.)

The method of undetermined coefficients proceeds as follows:

1. Solve the associated homogeneous problem, leaving the constants arbitrary. The general solution to the homogeneous problem is termed the "complementary solution."
2. Assume a "particular" solution as a weighted sum of all the elements of S. Leave the coefficients of the various terms "undetermined."
3. Sum the particular and complementary solutions, substitute into the original equation, and match the coefficients of like terms. This will determine values for some of the coefficients.
4. Impose the initial conditions on the resulting solution to determine the remaining coefficients.

Approach #3: Use the Laplace transform.

$$\mathcal{L}\{\dot{x} + ax = f(t)\} \quad \Rightarrow \quad (sX(s) - x_0) + aX(s) = F(s) \quad \Rightarrow \quad X(s) = \frac{x_0}{s+a} + \frac{1}{s+a}F(s).$$

Taking the inverse Laplace transform, we find

$$\begin{aligned} x(t) &= x_0 e^{-at} + e^{-at} * f(t) \\ &= x_0 e^{-at} + \int_0^t e^{-a(t-\tau)} f(\tau) d\tau. \end{aligned}$$

Compare this solution with (4).

Second order, homogeneous, LTI ODEs. We now turn our attention to the equation

$$\ddot{x} + a_1\dot{x} + a_0 x = 0, \qquad x(t_0) = x_0, \quad \dot{x}(t_0) = \dot{x}_0 \tag{5}$$

where a_1 and a_0 are scalar constants.

Approach: Again, assume that

$$x(t) = ce^{\lambda t} \tag{6}$$

Substituting the assumed form of $x(t)$ into the equation gives

$$\begin{aligned} 0 &= \lambda^2(ce^{\lambda t}) + a_1\lambda(ce^{\lambda t}) + a_0(ce^{\lambda t}) \\ &= (\lambda^2 + a_1\lambda + a_0)x(t). \end{aligned}$$

We assume the initial condition is such that $x(t) \neq 0$ for all time. Then it must be true that

$$\lambda^2 + a_1\lambda + a_0 = 0.$$

This quadratic equation has two distinct solutions, in general,

$$\lambda_{1,2} = \frac{1}{2}\left(-a_1 \pm \sqrt{a_1^2 - 4a_0}\right).$$

Thus, $x(t)$ generally contains two components of the form (6). By the principle of superposition, the general solution is a sum of the two:

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}. \tag{7}$$

Evaluating at the initial conditions determines the parameters c_1 and c_2. In the case that $a_1^2 - 4a_0 = 0$, the two candidate solutions are linearly dependent. In that case, we obtain a "secular" term in the solution:

$$x(t) = (c_1 + c_2 t)e^{-\frac{a_1}{2}t}.$$

Generically, the initial condition response of a second order system can be described either as exponential convergence or exponential divergence. The former case occurs whenever both eigenvalues have negative real part. The latter case occurs if *either* eigenvalue has a positive real part. The boundary which separates these two generic behaviors occurs when one or both of the eigenvalues lie on the imaginary axis. In this case, and assuming both eigenvalues are not zero, the initial condition response is a sinusoid whose frequency is the magnitude of the imaginary part.

If the initial condition response involves convergence to zero, then we say that the equilibrium at $(x, \dot{x}) = (0, 0)$ is *stable*. If the initial condition response involves divergence from zero, then we say that the equilibrium at $(x, \dot{x}) = (0, 0)$ is *unstable*. Otherwise, we say that the equilibrium at $(x, \dot{x}) = (0, 0)$ is *neutrally stable*.

We will be most concerned with the case where $a_0 > 0$ (which is necessary for "static stability"). Define the *natural frequency* ω_n and the *damping ratio* ζ as follows:

$$\omega_n = \sqrt{a_0} \quad \text{and} \quad \zeta = \frac{a_1}{2\sqrt{a_0}}.$$

With these definitions, the equation of motion becomes

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0, \qquad x(t_0) = x_0, \quad \dot{x}(t_0) = \dot{x}_0$$

and the characteristic equation is

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2\lambda = 0.$$

The characteristic values are

$$\begin{aligned} \lambda_{1,2} &= \frac{1}{2}\left(-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}\right) \\ &= \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n. \end{aligned}$$

Consider the following two cases:

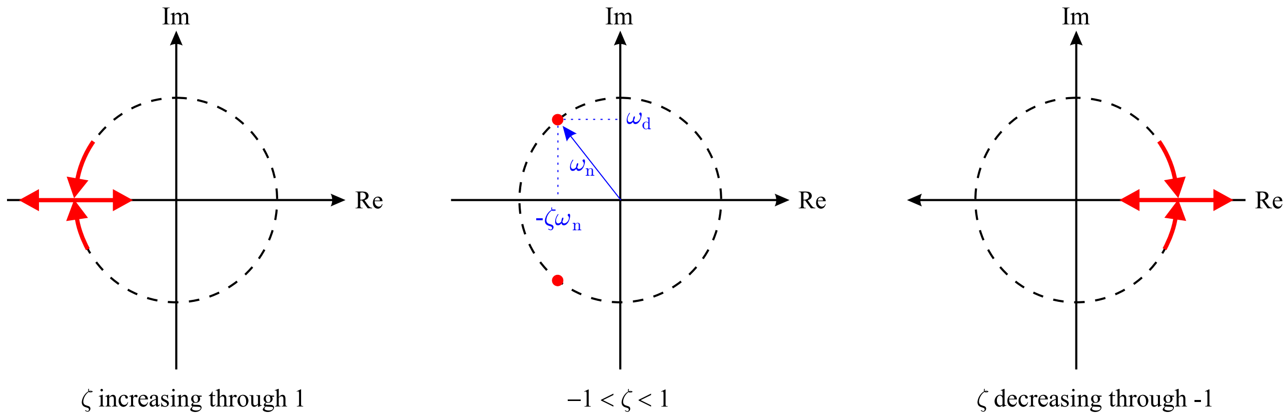

Figure 1: Characteristic Values of a Second Order System as the damping coefficient ζ varies.

1. $0 \leq \zeta^2 < 1$: In this case, the characteristic values are a complex conjugate pair. We may write

$$\begin{aligned}\lambda_{1,2} &= -\zeta\omega_n \pm j\sqrt{1-\zeta^2}\omega_n \\ &= -\zeta\omega_n \pm j\omega_d.\end{aligned}$$

   The initial condition response takes the form

$$x(t) = e^{-\zeta\omega_n t}\left(x_0 \cos\omega_d t + \frac{1}{\omega_d}\left(\dot{x}_0 + \zeta\omega_n x_0\right)\sin\omega_d t\right)$$

   If $\zeta > 0$, this response is a damped oscillation and we say that the equilibrium at the origin is dynamically stable, as well as statically stable. If $\zeta < 0$, the response diverges and we say that the equilibrium at the origin is dynamically unstable. If $\zeta = 0$, the response oscillates forever and we say that the equilibrium at the origin is neutrally stable.

   Referring to the factor $e^{-\zeta\omega_n t}$ in the expression above, the real part $-\zeta\omega_n$ plays a role similar to the reciprocal of a first order time constant. As in the case of a first order system, we may compute the time to half-amplitude (if $0 < \zeta < 1$) or the time to double amplitude (if $-1 < \zeta < 0$):

$$t_{\text{half}} \ \text{ or } \ t_{\text{double}} \approx \frac{0.69}{|\zeta|\omega_n}.$$

   Since the period of oscillation is $T_d = \frac{2\pi}{\omega_d}$, we may also compute the *number of oscillations to half or double amplitude* as

$$N_{\text{half/double}} = \frac{t_{\text{half/double}}}{T_d} \approx \left(\frac{0.69}{|\zeta|\omega_n}\right)\left(\frac{\omega_d}{2\pi}\right) = 0.11\sqrt{\left|\frac{1-\zeta^2}{\zeta^2}\right|}.$$

2. $\zeta^2 > 1$: In this case, the characteristic values are real. If $\zeta > 1$, then both characteristic values are negative. The response is an overdamped convergence to the equilibrium which is dominated by the larger (less negative) characteristic value. Thus, the response is essentially a first order response and one computes the time to half amplitude as one would for a first order system (using the larger characteristic value). As ζ grows larger positive, one characteristic value approaches zero from the left while the other moves toward $-\infty$. As the larger characteristic value approaches zero, convergence becomes slower and slower.

   If $\zeta < -1$, then both characteristic values are positive. The response is a non-oscillatory divergence which is dominated by the larger (more positive) characteristic value. One computes the time to double amplitude as for a first order system, using the larger characteristic value. As ζ grows larger negative, the dominant characteristic value moves toward ∞ and the other approaches zero from the right.

Second order, heterogeneous, LTI ODEs. Second order LTI ODEs with forcing can be solved using the method of undetermined coefficients or Laplace transforms exactly as in the case of first order ODEs.[1]

Higher order systems. We have seen that the the response of a first order system is (generically) exponential growth or decay. The response of a second order system is either a combination of first order responses (in the case that both characteristic values are real) or is an oscillatory exponential growth or decay (in the case that the characteristic values are complex conjugates). In fact, the roots of a polynomial with real coefficients are always real or complex conjugate pairs. Thus, with the exception of the "degenerate" case of repeated characteristic values, the general response of an n^{th} order linear system can be understood as a superposition of responses of first and second order responses.

[1]Even the technique of an integrating factor extends to second order equations, provided they are re-written as a system of first order equations; since this can always be done, the technique applies in general. The "generalized integrating factor" method is related to the method of "variation of parameters" which you may have seen in your course on differential equations.