Estimating Aerodynamic Properties

The aerodynamic properties of a given aircraft are critically dependent on the aerodynamic properties of its various lifting surfaces. For example, recall that the slope of the pitch moment coefficient C_{m_α}, a term of primary importance in determining longitudinal stability, takes the form

$$C_{m_\alpha} = C_{L_{\alpha_{\rm wb}}}\left(h - h_{n_{wb}}\right) - V_{\rm H} C_{L_{\alpha_t}}\left(1 - \frac{d\epsilon}{d\alpha}\right) + C_{m_{\alpha_p}}.$$

The value of this expression depends on $C_{L_{\alpha_{\rm wb}}}$, $C_{L_{\alpha_t}}$, $h_{n_{wb}}$, and $\frac{d\epsilon}{d\alpha}$, all of which are determined by the geometric properties of the wing and horizontal tail and parameters describing the flow field (such as the Mach number and Reynolds number). In this lecture, we discuss various tools available for estimating aerodynamic properties of lifting surfaces. These notes are adapted from [1].

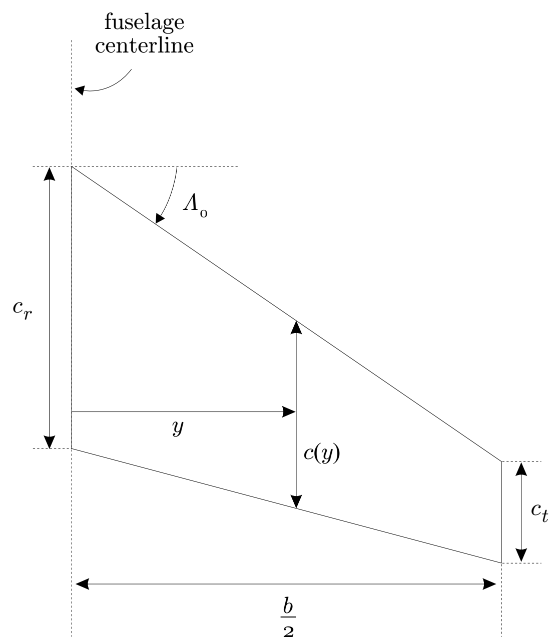

Figure 1: Sketch of a half-wing with constant sweep and taper.

Geometric properties of lifting surfaces. Consider the half-wing shown in Figure 1. Note that the wing has been extended to the centerline of the fuselage; the root chord c_r is defined there. The *mean geometric chord* of a wing is defined as

$$\begin{aligned} c' &= \frac{2}{b}\int_0^{\frac{b}{2}} c(y)dy \\ &= \frac{2}{b}\frac{S}{2} \end{aligned}$$

or

$$\boxed{c' = \frac{S}{b}.}$$

The wing aspect ratio is

$$\mathcal{R} = \frac{b}{c'} = \frac{b^2}{S}.$$

The *mean aerodynamic chord* of a wing is defined as

$$\bar{c} = \frac{2}{S}\int_0^{\frac{b}{2}} c(y)^2 dy.$$

(Note that this definition is purely geometric. The mean aerodynamic chord length does not depend on the wing aerodynamics.)

For the wing depicted in Figure 1, the local chord $c(y)$ varies linearly from the root chord c_r to the tip chord c_t, so we say that this wing has *constant taper*. We define the *taper ratio*

$$\lambda = \frac{c_t}{c_r}.$$

For a wing with constant taper, the mean geometric chord is

$$c' = \left(\frac{1+\lambda}{2}\right) c_r$$

and the mean aerodynamic chord is

$$\bar{c} = \frac{2}{3}\left(\frac{1+\lambda+\lambda^2}{1+\lambda}\right) c_r.$$

Note that, if $\lambda = 1$ (i.e., if the wing is untapered), then $c' = \bar{c} = c_r = c_t$.

The n^{th} *percent chord line* of a wing is the curve connecting each point $nc(y)$ measured aft of the local leading edge (where $0 \leq n \leq 1$). (For example, the 0^{th} percent chord line is the leading edge of the wing.) For the wing depicted, the angle of incidence of each chord line is constant, so we say that this wing has *constant sweep*. For a wing with constant sweep, one defines the *sweep angle* Λ_n of the n^{th} percent chord line as shown for $n = 0$. Given the sweep angle Λ_m of the m^{th} percent chord line for a wing with constant sweep and taper, one may determine Λ_n according to the formula

$$\tan\Lambda_n = \tan\Lambda_m - \frac{4(n-m)}{\mathcal{R}}\left(\frac{1-\lambda}{1+\lambda}\right).$$

The *mean aerodynamic center* $(\bar{x}, \bar{y}, \bar{z})$ is defined as that point about which the total aerodynamic moment of the wing does not vary with angle of attack. The location of the mean aerodynamic chord (for the half-wing) is determined relative to the mean aerodynamic center. While the mean aerodynamic chord $\bar{c}$ is defined purely by the wing geometry, the location of the mean aerodynamic center depends on the wing loading. Under certain assumptions about the form of the load distribution, the location of the mean aerodynamic center may be computed explicitly. See Appendix C of [1] and, in particular, Figure C.3.

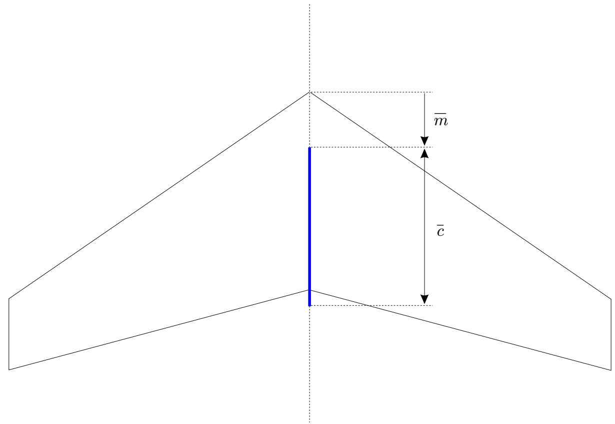

Figure 2: Sketch of the mean aerodynamic chord for a wing with constant sweep and taper.

Because the airplane is symmetric about the vertical plane through the centerline, the mean aerodynamic center of the complete wing lies in this plane. For a wing with constant sweep and taper whose load distribution is proportional to the local chord length, the distance from the leading edge of the root chord to the leading edge of the mean aerodynamic chord is

$$\boxed{\bar{m} = \frac{b}{6}\left(\frac{1+2\lambda}{1+\lambda}\right)\tan\Lambda_0.}$$

Such a load distribution would result, for example, if the local lift coefficient of the airfoil sections constituting the wing does not vary with y. (See Appendix C of [1] for information on other load distributions and planform shapes.)

Aerodynamic properties of airfoils (Sectional properties). The aerodynamic properties of lifting surfaces can be estimated in terms of the aerodynamic properties of the 2-D airfoil sections which make up the complete lifting surface. For a given lifting surface we generally need three 2-D properties

C_{l_α} = the 2-D lift-curve slope

$\alpha_{0L_{2D}}$ = the 2-D zero-lift angle of attack, and

$C_{m_{0L_{2D}}}$ = the 2-D pitch moment at zero lift (i.e., the moment coefficient about the aerodynamic center).

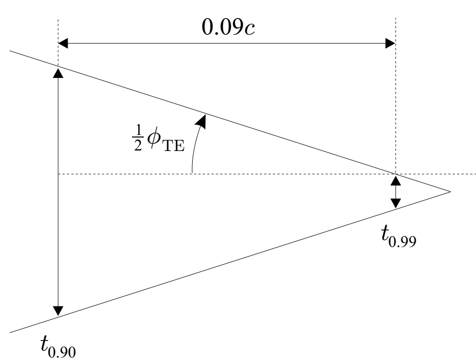

Figure 3: Definition of the trailing edge angle ϕ_{TE}.

We first consider the problem of estimating the 2D lift-curve slope C_{l_α} for a given section. The method outlined in Section 1 of Appendix B in [1] applies to wings with constant sweep and taper and without twist moving at Mach numbers less than the critical Mach number. (The critical Mach number is that value of the free stream Mach number at which the local flow is sonic at some point on the aircraft; the critical Mach number is always less than one.) The method proceeds as follows

1. Given the wing thickness ratio $\frac{t}{c}$, estimate the theoretical 2-D lift-curve slope $(C_{l_\alpha})_{\text{theory}}$ from Figure B.1,1(b). The curve appears to be well-approximated by the formula

$$\boxed{(C_{l_\alpha})_{\text{theory}} = 2\pi + 4.9\frac{t}{c}.}$$

2. Measure or estimate the *trailing edge angle* ϕ_{TE}. This angle is defined

$$\boxed{\phi_{\text{TE}} = 2\arctan\left(\frac{\frac{1}{2}\left(\frac{t}{c}\right)_{90\%} - \frac{1}{2}\left(\frac{t}{c}\right)_{99\%}}{0.09}\right)}$$

   where $\left(\frac{t}{c}\right)_{90\%}$ is the wing thickness ratio at the 90% chord line and $\left(\frac{t}{c}\right)_{99\%}$ is the wing thickness ratio at the 99% chord line. See Figure 3.

3. Determine the factor K which corrects for trailing edge angle and Reynolds number from Figure B.1,1(a). (Interpolate for Reynolds numbers between 10^6 and 10^8.)

4. Correct for trailing edge angle, Reynolds number, and Mach number to obtain the 2-D lift-curve slope

$$\boxed{C_{l_\alpha} = \frac{1.05}{\sqrt{1-M^2}} K\,(C_{l_\alpha})_{\text{theory}}.}$$

If detailed airfoil information is unavailable and the wing is *thin*, then an approximate value of the 2-D lift-curve slope is

$$\boxed{C_{l_\alpha} = \frac{2\pi}{\sqrt{1-M^2}}.}$$

The parameters $\alpha_{0L_{2D}}$ and $C_{m_{0L_{2D}}}$ must be determined by some other means, such as a wind tunnel test or a computational fluid dynamics (CFD) model.

Aerodynamic properties of lifting surfaces. First, we determine the lift-curve slope for an untwisted wing with constant sweep and taper moving at a subcritical Mach number. Let

$$\boxed{\kappa = \frac{\sqrt{1-M^2}}{2\pi} C_{l_\alpha}.}$$

Note that $\kappa = 1$ for a thin wing (i.e., a flat plate). Using the sectional lift-curve slope from above, and correcting for the finite wing-span and the sweep angle, we obtain

$$C_{L_\alpha} = \frac{2\pi A\!\!R}{2 + \sqrt{\left(\frac{A\!\!R^2(1-M^2)}{\kappa^2}\right)\left(1 + \frac{\tan^2\Lambda_{1/2}}{(1-M^2)}\right) + 4}}.$$

Next, we determine the zero-lift angle of attack α_{0L}. We assume that the sectional zero-lift angle $\alpha_{0L_{2D}}$ is given. Following are two cases for which α_{0L} can be easily determined.

1. Constant sweep angle and constant airfoil section, where the cross sections are taken normal to the n^{th} percent chord line. (No twist.)

$$\alpha_{0L} = \arctan\left(\frac{\tan\alpha_{0L_{2D}}}{\cos\Lambda_n}\right).$$

   Note that, in the special case of an unswept wing, $\alpha_{0L} = \alpha_{0L_{2D}}$.

2. Geometric twist and zero sweep. (Note: A wing with *geometric twist* has constant airfoil sections across the span, however the zero-lift line of the sections varies from root to tip. A wing with *aerodynamic twist* achieves a similar effect by varying the airfoil section from root to tip.) Refer the wing angle of attack to the root chord: $\alpha_w := \alpha_{\text{root}}$. Then

$$\alpha(y) = \alpha_{\text{root}} + \Theta(y)$$

   where $\Theta(y)$ is the *wing twist* and $\Theta(0) = 0$. The wing zero-lift angle of attack is

$$\alpha_{0L} = \frac{2}{S}\int_0^{b/2} \left(\alpha_{0L_{2D}} - \Theta(y)\right) c(y) dy.$$

We next compute the wing zero-lift pitch moment $C_{m_{0L}}$, given $C_{m_{0L_{2D}}}$ across the span. This term can be easily estimated in the special case of an untwisted wing with constant sweep angle, where the airfoil sections are taken parallel to the free stream:

$$C_{m_{0L}} = \left(\frac{A\!\!R\cos^2\Lambda_{1/4}}{A\!\!R + 2\cos\Lambda_{1/4}}\right)\left(\frac{C_{m_{0L_{2D}}}|_{\text{root}} + C_{m_{0L_{2D}}}|_{\text{tip}}}{2}\right).$$

Note that, if the wing section does not vary across the span, the latter factor becomes simply $C_{m_{0L_{2D}}}$.

Wing downwash parameter $\frac{d\epsilon}{d\alpha}$. The wing downwash parameter can be crudely estimated by assuming a thin, finite wing with an elliptic load distribution, which gives

$$\frac{d\epsilon}{d\alpha} = \frac{2C_{L_\alpha}}{\pi A\!\!R}.$$

A more accurate method applicable to wings with constant sweep and taper is described in Appendix B.5 of [1]. The estimate takes the form

$$\frac{d\epsilon}{d\alpha} = \frac{4.44\left[K_{\text{A}}K_\lambda K_{\text{H}}\sqrt{\cos\Lambda_{1/4}}\right]^{1.19}}{\sqrt{1-M^2}}$$

where K_{A} is a correction for the aspect ratio of the wing, K_λ is a correction for the taper ratio of the wing, and K_{H} is a correction for the location of the horizontal tail. The correction factors are determined according to the following formulas:

$$\begin{aligned} K_{\mathrm{A}} &= \frac{1}{AR} - \frac{1}{1+AR^{1.7}} \\ K_\lambda &= \frac{10-3\lambda}{7} \\ K_{\mathrm{H}} &= \frac{1-\left|\frac{h_H}{b}\right|}{\sqrt[3]{\frac{2l_H}{b}}} \end{aligned}$$

where, in the last formula,

h_H = the (signed) orthogonal distance from the extended root chord line to the horizontal tail a.c.

l_H = the longitudinal distance from the wing a.c. to the tail a.c.

b = wingspan.

Tutorial Example

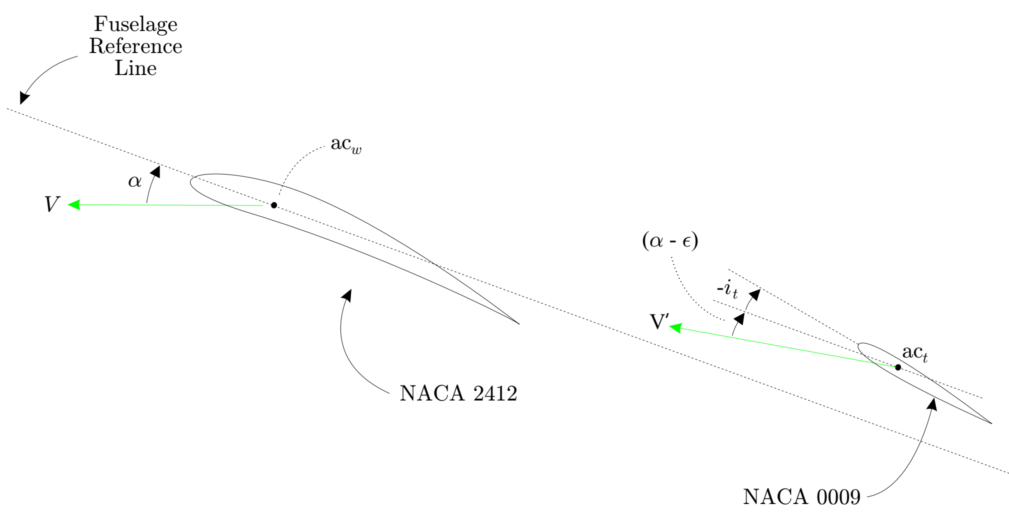

Figure 4: Sketch for the example.

Consider an aircraft with a trapezoidal wing and horizontal tail. Following are the geometric properties.

Wing properties: NACA 2412

$$b = 15 \text{ m}, \qquad AR = 6, \qquad \Lambda_{1/4} = \frac{\pi}{6} \text{ rad}, \qquad \lambda = \frac{1}{4},$$

$$\alpha_{0L_{2\mathrm{D}}} = -2.0^\circ, \qquad c_{m_{0L_{2\mathrm{D}}}} = -0.047.$$

Tail properties: NACA 0009

$$b_t = 6 \text{ m}, \qquad AR_t = 4, \qquad \Lambda_{1/4_t} = 0 \text{ rad}, \qquad \lambda_t = 1,$$

$$\alpha_{0L_{2\mathrm{D}t}} = 0.0^\circ, \qquad c_{m_{0L_{2\mathrm{D}t}}} = 0.0, \qquad i_t = 1.0^\circ, \qquad h_H = 0 \text{ m}, \qquad l_H = 15 \text{ m}.$$

Compute the following:

1. The mean aerodynamic chord length and location for the wing and tail and the longitudinal location of the wing and tail mean aerodynamic centers.
2. The lift coefficient of the wing as a function of fuselage angle of attack α at $Re = 10^6$ and $M = 0.5$. (This requires computing both the slope of the lift curve and the lift coefficient at $\alpha = 0$.)
3. The zero-lift pitch moment of the wing $C_{m_{0L_w}} = C_{m_{\mathrm{ac}w}}$.
4. The lift coefficient of the tail as a function of fuselage angle of attack α at $Re = 10^6$ and $M = 0.5$. (In addition to the lift slope, this requires computing the downwash parameter $\frac{d\epsilon}{d\alpha}$.)

Item #1.)

Wing m. a. chord length. The mean *geometric* chord is

$$c' = \frac{S}{b} = \frac{b}{AR} = 2.5 \text{ m}.$$

Because the wing is trapezoidal, we compute

$$c_r = \frac{2}{1+\lambda} c' = 4 \text{ m}.$$

The mean *aerodynamic* chord is

$$\bar{c} = \frac{2}{3}\left(\frac{1+\lambda+\lambda^2}{1+\lambda}\right) c_r = 2.8 \text{ m}.$$

Wing m. a. chord location. Assuming a uniform load distribution, the distance from the leading edge of the root chord to the leading edge of the mean aerodynamic chord is

$$\bar{m} = \frac{b}{6}\left(\frac{1+2\lambda}{1+\lambda}\right) \tan\Lambda_0$$

where

$$\tan\Lambda_0 = \tan\Lambda_{1/4} - \frac{4(0-\frac{1}{4})}{AR}\left(\frac{1-\lambda}{1+\lambda}\right) = 0.68$$

which corresponds to a leading edge sweep angle of about 34°. We thus compute

$$\bar{m} = \frac{b}{6}\left(\frac{1+2\lambda}{1+\lambda}\right) \tan\Lambda_0 = 2.03 \text{ m}.$$

The leading edge of the mean aerodynamic chord is roughly two meters aft of the wing apex.

Wing m. a. center. We may estimate the location of the wing mean aerodynamic center from Figure C.3 in [1]. For a taper ratio $\lambda = 0$ and aspect ratio $AR = 6$, we would find that $\bar{x}_w \approx 0.32\bar{c}$. For a taper ratio $\lambda = \frac{1}{2}$ and aspect ratio $AR = 6$, we would find that $\bar{x}_w \approx 0.26\bar{c}$. Interpolating for $\lambda = \frac{1}{4}$, we find that

$$\bar{x}_w \approx 0.29\bar{c} = 0.81 \text{ m}$$

aft of the leading edge of the mean aerodynamic chord, which means the wing aerodynamic center is 2.84 meters aft of the wing apex.

Tail m. a. chord length and location. The mean geometric chord of the tail is

$$c'_t = \frac{S_t}{b_t} = \frac{b_t}{A\!\!R_t} = 1.5 \text{ m}.$$

Because the wing is not tapered ($\lambda = 1$),

$$\bar{c}_t = c'_t = 1.5 \text{ m}.$$

Tail m. a. center. From Figure C.3 in [1], we estimate that the tail mean aerodynamic center is located at

$$\bar{x}_t \approx 0.20\bar{c} = 0.30 \text{ m}$$

aft of the leading edge of the tail.

Item #2.)

Wing lift-curve slope. Based on the definition of the NACA 4-digit airfoil series, the thickness ratio for the NACA 2412 is $\frac{t}{c} = 0.12$. (See [2] for details about the definitions of the 4, 5, and 6 digit series.) From Figure B.1,1 (b) in [1], we find that

$$(C_{l_\alpha})_{\text{theory}} \approx 6.87 \text{ rad}^{-1}.$$

Referring again to the definition of the NACA 4-digit series, one may compute

$$\phi_{\text{TE}_w} = 2 \arctan\left(\frac{\frac{1}{2}\left(\frac{t}{c}\right)_{90\%} - \frac{1}{2}\left(\frac{t}{c}\right)_{99\%}}{0.09}\right) = 0.26 \text{ rad}$$

which is around 15°. From Figure B.1,1 (a), we find that

$$K \approx 0.77.$$

The 2-D lift-curve slope, corrected for wing thickness, trailing edge angle, Reynolds number and Mach number, is

$$C_{l_\alpha} = \frac{1.05}{\sqrt{1-M^2}} K \, (C_{l_\alpha})_{\text{theory}} \approx 6.41 \text{ rad}^{-1}.$$

We next compute

$$\kappa = \frac{\sqrt{1-M^2}}{2\pi} C_{l_\alpha} = 0.88.$$

The formula for the lift-curve slope $C_{L_{\alpha_w}}$ requires the sweep angle of the mid-chord line. We compute

$$\Lambda_{1/2} = \arctan\left\{\tan\Lambda_{1/4} - \frac{4(\frac{1}{2}-\frac{1}{4})}{A\!\!R}\left(\frac{1-\lambda}{1+\lambda}\right)\right\} = 0.45 \text{ rad}$$

or about 26°. We thus find that

$$C_{L_{\alpha_w}} = \frac{2\pi A\!\!R}{2 + \sqrt{\left(\frac{A\!\!R^2(1-M^2)}{\kappa^2}\right)\left(1 + \frac{\tan^2\Lambda_{1/2}}{(1-M^2)}\right) + 4}} = 4.2 \text{ rad}^{-1}$$

or about 0.07 per degree.

Wing zero-lift angle of attack. Now, the zero-lift angle of attack of the airfoil is given to be $\alpha_{0L_{2\text{D}}} = -2.0°$. Thus, the zero-lift angle of attack of the wing is

$$\alpha_{0L_w} = \arctan\left(\frac{\tan\alpha_{0L_{2\text{D}}}}{\cos\Lambda_{1/4}}\right) = -0.040 \text{ rad}$$

or about $-2.3°$. We thus find that

$$\begin{aligned} C_{L_w} &= C^{\circ}_{L_{\alpha_w}}(\alpha_w - \alpha_{0L_w}) \\ &= C^{\circ}_{L_{\alpha_w}}(\alpha - \alpha_{0L_w}) \\ &= 0.07(\alpha - (-2.3)) \\ &= 0.16 + 0.07\alpha, \end{aligned}$$

where $C^{\circ}_{L_{\alpha_w}}$ has units of $\deg^{-1}$ and α is given in degrees.

Item #3.)

Using the formula for $C_{m_{0L}}$ presented earlier, we find that

$$C_{m_{0L_w}} = \left(\frac{A\!\!R \cos^2 \Lambda_{1/4}}{A\!\!R + 2\cos\Lambda_{1/4}}\right) C_{m_{0L_{2D}}} = -0.027.$$

Item #4.)

Tail lift-curve slope. The tail lift coefficient is

$$\begin{aligned} C_{L_t} &= C_{L_{\alpha_t}}(\alpha - \epsilon(\alpha_w) - i_t) \\ &= C_{L_{\alpha_t}}\left(\alpha - \left(\epsilon_0 + \frac{d\epsilon}{d\alpha}\alpha\right) - i_t\right) \\ &= -\left\{C_{L_{\alpha_t}}(\epsilon_0 + i_t)\right\} + \left\{C_{L_{\alpha_t}}\left(1 - \frac{d\epsilon}{d\alpha}\right)\right\}\alpha. \end{aligned}$$

The critical parameters which we must compute are a_t, ϵ_0, and $\frac{d\epsilon}{d\alpha}$.

Based on the definition of the NACA 4-digit airfoil series, the thickness ratio is $\frac{t}{c} = 0.09$. From Figure B.1,1 (b) in [1], we find that

$$(C_{l_\alpha})_{\text{theory}} \approx 6.75 \text{ rad}^{-1}.$$

(The subscript t is omitted with the understanding that all of the following computations relate to the horizontal tail.) Referring again to the definition of the NACA 4-digit series, one may compute

$$\phi_{\text{TE}_t} = 2\arctan\left(\frac{\frac{1}{2}\left(\frac{t}{c}\right)_{90\%} - \frac{1}{2}\left(\frac{t}{c}\right)_{99\%}}{0.09}\right) = 0.20 \text{ rad}$$

which is around $11°$. From Figure B.1,1 (a), we find that

$$K \approx 0.79.$$

The 2-D lift slope, corrected for wing thickness, trailing edge angle, Reynolds number and Mach number, is

$$C_{l_\alpha} = \frac{1.05}{\sqrt{1-M^2}} K\,(C_{l_\alpha})_{\text{theory}} \approx 6.47 \text{ rad}^{-1}.$$

We next compute

$$\kappa = \frac{\sqrt{1-M^2}}{2\pi} C_{l_\alpha} = 0.89.$$

Because the horizontal tail is unswept, we compute

$$\begin{aligned} C_{L_{\alpha_t}} &= \frac{2\pi A\!\!R}{2+\sqrt{\left(\frac{A\!\!R^2(1-M^2)}{\kappa^2}\right)\left(1+\frac{\tan^2\Lambda_{1/2}}{(1-M^2)}\right)+4}} \\ &= \frac{2\pi A\!\!R}{2+\sqrt{\left(\frac{A\!\!R^2(1-M^2)}{\kappa^2}\right)+4}} \\ &= 3.9\ \text{rad}^{-1} \end{aligned}$$

or about 0.07 per degree. Since the horizontal tail is symmetric, the zero-lift angle of attack of the airfoil, and of the entire tail, is zero.

Downwash at the tail. Next, we estimate the downwash parameter $\frac{d\epsilon}{d\alpha}$ using the relation

$$\frac{d\epsilon}{d\alpha} = \frac{4.44\left[K_\text{A} K_\lambda K_\text{H}\sqrt{\cos\Lambda_{1/4}}\right]^{1.19}}{\sqrt{1-M^2}}.$$

The correction factors are:

$$\begin{aligned} K_\text{A} &= \frac{1}{A\!\!R} - \frac{1}{1+A\!\!R^{1.7}} = 0.12 \\ K_\lambda &= \frac{10-3\lambda}{7} = 1.32 \\ K_\text{H} &= \frac{1-\left|\frac{h_H}{b}\right|}{\sqrt[3]{\frac{2l_H}{b}}} = 0.79. \end{aligned}$$

We thus compute

$$\frac{d\epsilon}{d\alpha} = 0.40.$$

To determine ϵ_0, we observe that no downwash is generated when the wing generates no lift. The zero-lift angle of attack of the wing is $\alpha_{0L_w} = -2.3°$. We thus compute

$$\begin{aligned} \epsilon(\alpha_w) = 0 &= \epsilon_0 + \frac{d\epsilon}{d\alpha}\alpha_{0L_w} \\ &= \epsilon_0 + \frac{d\epsilon}{d\alpha}(-2.3°) \end{aligned}$$

which tells us that $\epsilon_0 = 0.92° = 0.016$ rad.

In the end, we obtain

$$\begin{aligned} C_{L_t} &= -\left\{C_{L_{\alpha_t}}(\epsilon_0 + i_t)\right\} + \left\{C_{L_{\alpha_t}}\left(1-\frac{d\epsilon}{d\alpha}\right)\right\}\alpha \\ &= -0.14 + 0.04\alpha \end{aligned}$$

where α is measured in degrees. Notice that $C_{L_t} < 0$ when $\alpha = 0$. At zero angle of attack, the tail generates a downward force which generates a nose-up moment about the center of gravity, as desired.

References

[1] B. Etkin and L. D. Reid. *Dynamics of Flight: Stability and Control.* John Wiley and Sons, New York, NY, third edition, 1996.

[2] W. H. Mason. Geometry for aerodynamicists. Notes available on-line at: `http://www.aoe.vt.edu/aoe/faculty/Mason_f/CAtxtAppA.pdf`.