Lecture 9: Lateral-Directional Steady Flight

A complete discussion of lateral-directional equilibrium flight requires some advanced material which we will cover in coming lectures. For this reason, the authors of [1] postpone the topic until Section 7.8. There is no great difficulty in studying lateral-directional equilibrium flight, however, if one is willing to temporarily accept a few observations without proof.

As we will see shortly, the attitude kinematic equations for a rigid aircraft are

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}. \tag{1}$$

In considering lateral-directional steady flight, we will assume that ϕ and θ remain constant. For most flight conditions, the 3×3 matrix above is invertible. Moreover, if ϕ and θ remain constant,

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta\sin\phi \\ \cos\theta\cos\phi \end{pmatrix} \dot{\psi}. \tag{2}$$

If the aircraft is not turning, then $\dot{\psi} = 0$ and $\boldsymbol{\omega} = \mathbf{0}$. Otherwise, $\boldsymbol{\omega}$ is a constant vector which is parallel to the *inertial* vertical axis.

As we will soon derive, the dynamic equations are

$$m\dot{\boldsymbol{v}} = m\boldsymbol{v} \times \boldsymbol{\omega} + \begin{pmatrix} X(\boldsymbol{v},\boldsymbol{\omega},\boldsymbol{u}) \\ Y(\boldsymbol{v},\boldsymbol{\omega},\boldsymbol{u}) \\ Z(\boldsymbol{v},\boldsymbol{\omega},\boldsymbol{u}) \end{pmatrix} + W \begin{pmatrix} -\sin\theta \\ \cos\theta\sin\phi \\ \cos\theta\cos\phi \end{pmatrix} \tag{3}$$

$$\boldsymbol{I}\dot{\boldsymbol{\omega}} = \boldsymbol{I}\boldsymbol{\omega} \times \boldsymbol{\omega} + \begin{pmatrix} L(\boldsymbol{v},\boldsymbol{\omega},\boldsymbol{u}) \\ M(\boldsymbol{v},\boldsymbol{\omega},\boldsymbol{u}) \\ N(\boldsymbol{v},\boldsymbol{\omega},\boldsymbol{u}) \end{pmatrix}. \tag{4}$$

For steady flight, $\dot{\boldsymbol{v}} = \mathbf{0}$ and $\dot{\boldsymbol{\omega}} = \mathbf{0}$. Moreover, in normal flight, all components of $\boldsymbol{v}$ and $\boldsymbol{\omega}$ are small, with the exception of $u \approx V$. Arguing (informally, for now) that products of small terms are negligible we may write

$$m\boldsymbol{v} \times \boldsymbol{\omega} \approx \begin{pmatrix} 0 \\ -mVr \\ mVq \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{I}\boldsymbol{\omega} \times \boldsymbol{\omega} \approx \mathbf{0}.$$

Thus, for steady flight, we require

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 \\ mVr \\ -mVq \end{pmatrix} + W \begin{pmatrix} \sin\theta \\ -\cos\theta\sin\phi \\ -\cos\theta\cos\phi \end{pmatrix}$$

$$\begin{pmatrix} L \\ M \\ N \end{pmatrix} = \mathbf{0}.$$

We will assume, in the following, that the pitch angle θ has been re-defined so as to be zero in horizontal flight at the nominal speed. This assumption defines an alternate body-fixed reference frame, which we will discuss shortly, referred to as the *stability reference frame*. In equilibrium flight conditions for which $\theta \neq 0$, the aircraft is climbing or descending so the corresponding constant value of θ is referred to as the *climb angle*.

Non-turning flight with steady sideslip. For steady, non-turning flight, $\dot{\psi} = 0$ which implies that $q = r = 0$. In this case, the *lateral-directional* force and moment equations are

$$\begin{aligned} Y &\approx -W \sin\phi \\ L &= 0 \\ N &= 0. \end{aligned}$$

For small roll angles, we may write

$$\begin{aligned} C_{Y_\beta}\beta + C_{Y_{\delta a}}\delta a + C_{Y_{\delta r}}\delta r &\approx -C_W \phi \\ C_{l_\beta}\beta + C_{l_{\delta a}}\delta a + C_{l_{\delta r}}\delta r &= 0 \\ C_{n_\beta}\beta + C_{n_{\delta a}}\delta a + C_{n_{\delta r}}\delta r &= 0 \end{aligned}$$

where $C_W = \frac{W}{\left(\frac{1}{2}\rho V^2\right)S}$ is the wing loading. These are three linear algebraic equations in the four unknowns β, δa, δr and ϕ. The system is under-determined. To solve the equations, we fix one value to obtain three equations in three unknowns.

Scenario #1: Landing in a cross-wind. For example, suppose one wishes to land in a cross-wind. To align the aircraft with the runway just before touchdown requires flying at a steady sideslip angle

$$\beta = \sin^{-1}\left(\frac{\text{Crosswind velocity}}{\text{Total airspeed}}\right).$$

Then one would solve the equations

$$\begin{pmatrix} C_{Y_{\delta a}} & C_{Y_{\delta r}} & C_W \\ C_{l_{\delta a}} & C_{l_{\delta r}} & 0 \\ C_{n_{\delta a}} & C_{n_{\delta r}} & 0 \end{pmatrix} \begin{pmatrix} \delta a \\ \delta r \\ \phi \end{pmatrix} = -\begin{pmatrix} C_{Y_\beta} \\ C_{l_\beta} \\ C_{n_\beta} \end{pmatrix} \beta$$

to obtain the necessary control commands and the corresponding roll angle.

Example. As an example, let's consider the Navion, whose mass and geometric parameters are

$$W = 2750 \text{ lbs}, \qquad S = 184 \text{ ft}^2, \qquad b = 33.4 \text{ ft}.$$

For sea level flight at $M = 0.158$, we have the following stability derivatives (all in units of "per radian"):

	$C_{Y_{(\cdot)}}$	$C_{l_{(\cdot)}}$	$C_{n_{(\cdot)}}$
β	-0.564	-0.074	0.071
δa	0	0.134	-0.0035
δr	0.157	0.107	-0.072

The density is $\rho = 2.3769 \times 10^-3$ slugs/ft^3 and the airspeed is $V = 176$ ft/s. The weight coefficient is therefore

$$C_W = \frac{2750}{\left(\frac{1}{2}(2.3769E-3)(176)^2\right)(184)} = 0.406.$$

Suppose one wishes to land in a 40 ft/s cross-wind; in this case,

$$\beta = \arcsin\left(\frac{40}{176}\right) = 0.223 \text{ rad} \approx 13.1^\circ.$$

Solving

$$\begin{pmatrix} C_{Y_{\delta a}} & C_{Y_{\delta r}} & C_W \\ C_{l_{\delta a}} & C_{l_{\delta r}} & 0 \\ C_{n_{\delta a}} & C_{n_{\delta r}} & 0 \end{pmatrix} \begin{pmatrix} \delta a \\ \delta r \\ \phi \end{pmatrix} = - \begin{pmatrix} C_{Y_\beta} \\ C_{l_\beta} \\ C_{n_\beta} \end{pmatrix} \beta$$

gives

$$\begin{pmatrix} \delta a \\ \delta r \\ \phi \end{pmatrix} = \begin{pmatrix} 0.0253 \\ 0.109 \\ 0.118 \end{pmatrix} \text{rad} \approx \begin{pmatrix} 1.5^\circ \\ 6.2^\circ \\ 6.7^\circ \end{pmatrix}$$

Scenario #2: Maximum allowable cross-wind at landing. Alternatively, suppose one wishes to compute the maximum crosswind (equivalently, the maximum sideslip angle β) in which an airplane can land. Assuming that maximum rudder is the limiting condition, one would solve

$$\begin{pmatrix} C_{Y_\beta} & C_{Y_{\delta a}} & C_W \\ C_{l_\beta} & C_{l_{\delta a}} & 0 \\ C_{n_\beta} & C_{n_{\delta a}} & 0 \end{pmatrix} \begin{pmatrix} \beta \\ \delta a \\ \phi \end{pmatrix} = - \begin{pmatrix} C_{Y_{\delta r}} \\ C_{l_{\delta r}} \\ C_{n_{\delta r}} \end{pmatrix} (\delta r)_{\max}.$$

If the resulting aileron deflection δa is larger than the maximum allowable deflection, then the aileron is the limiting factor and one must solve for the corresponding sideslip angle, rudder angle, and roll angle.

Scenario #3: Control conditions for asymmetric thrust. One may also use the given equations to determine the complete control conditions for asymmetric thrust. In this case, the original equations become

$$\begin{aligned} C_{Y_\beta}\beta + C_{Y_{\delta a}}\delta a + C_{Y_{\delta r}}\delta r &= -C_W\phi \\ C_{l_\beta}\beta + C_{l_{\delta a}}\delta a + C_{l_{\delta r}}\delta r &= 0 \\ C_{n_\beta}\beta + C_{n_{\delta a}}\delta a + C_{n_{\delta r}}\delta r &= -C_{n_T} \end{aligned}$$

where, as was mentioned in a previous lecture,

$$C_{n_T} = -\frac{T y_p}{\left(\frac{1}{2}\rho V^2\right) Sb} = -C_T \frac{S_p y_p}{Sb}.$$

Requiring that the sideslip angle be zero, we obtain

$$\begin{pmatrix} C_{Y_{\delta a}} & C_{Y_{\delta r}} & C_W \\ C_{l_{\delta a}} & C_{l_{\delta r}} & 0 \\ C_{n_{\delta a}} & C_{n_{\delta r}} & 0 \end{pmatrix} \begin{pmatrix} \delta a \\ \delta r \\ \phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -C_{n_T} \end{pmatrix}.$$

If either the resulting rudder angle or the resulting aileron angle is beyond the capability of the actuator, then the actuators must be modified to provide more control power.

Steady turning flight revisited. Recall that we have already discussed the longitudinal conditions for a steady turn at constant radius. The essential difference between this condition and wings level (non-slipping) flight is that the lift vector is deflected inward. The vertical component of lift must balance the airplane's weight while the horizontal component provides the centripetal acceleration necessary to maintain the turn.

Per the discussion in Section 7.8 of [1], we define a "truly banked turn" (or a "coordinated turn") as one for which (1) the angular velocity vector is constant and vertical (in the inertial frame) and (2) the resultant of gravity and centrifugal force lies in the plane of symmetry.[1] Equivalently, it is a turn for which the

[1] Recall that "centrifugal force" is a fictitious force which explains the feeling of being 'flung outward' when experiencing a turn. It is the negative of the (very real) force which is necessary to maintain the turn.

lateral aerodynamic force Y is identically zero. Thus, in a truly banked turn, the pilot and passengers will feel the combination of their own weight and their inward acceleration through the seat of their pants.

Because $Y = 0$ for a truly banked turn, we have

$$\begin{aligned} W \cos\theta \sin\phi &= mVr \\ &= mV \left(\dot{\psi} \cos\theta \cos\phi \right) \end{aligned}$$

from which we obtain ϕ in terms of turn rate $\dot{\psi}$:

$$\tan\phi = \frac{V}{g} \dot{\psi}.$$

For a steady turn at constant altitude,

$$\tan\phi = \frac{V^2}{g R_{\text{turn}}}.$$

In this case, ϕ may be specified by the desired speed and the desired turn radius.

The non-dimensional lateral-directional force and moment equations for steady turning flight are

$$\begin{aligned} C_{Y_\beta}\beta + C_{Y_p}\hat{p} + C_{Y_r}\hat{r} + C_{Y_{\delta a}}\delta a + C_{Y_{\delta r}}\delta r &= \frac{1}{\left(\frac{1}{2}\rho V^2\right) S} \left(-W \cos\theta \sin\phi + mVr \right) \\ C_{l_\beta}\beta + C_{l_p}\hat{p} + C_{l_r}\hat{r} + C_{l_{\delta a}}\delta a + C_{l_{\delta r}}\delta r &= 0 \\ C_{n_\beta}\beta + C_{n_p}\hat{p} + C_{n_r}\hat{r} + C_{n_{\delta a}}\delta a + C_{n_{\delta r}}\delta r &= 0 \end{aligned}$$

where $\hat{p}$ and $\hat{r}$ represent nondimensional roll and pitch rate, respectively:

$$\hat{p} = \frac{b}{2V} p \quad \text{and} \quad \hat{r} = \frac{b}{2V} r.$$

Recalling that $W \cos\theta \sin\phi = mVr$ for a truly banked turn, the right hand side of the Y coefficient equation vanishes. Recalling equation (2), we have:

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} \approx \begin{pmatrix} -\sin\theta \\ \cos\theta \sin\phi \\ \cos\theta \cos\phi \end{pmatrix} \dot{\psi}.$$

Substituting into the definitions $\hat{p}$ and $\hat{r}$ and rearranging the steady flight equations above gives

$$\begin{pmatrix} C_{Y_\beta} & C_{Y_{\delta a}} & C_{Y_{\delta r}} \\ C_{l_\beta} & C_{l_{\delta a}} & C_{l_{\delta r}} \\ C_{n_\beta} & C_{n_{\delta a}} & C_{n_{\delta r}} \end{pmatrix} \begin{pmatrix} \beta \\ \delta a \\ \delta r \end{pmatrix} = \begin{pmatrix} C_{Y_p} & C_{Y_r} \\ C_{l_p} & C_{l_r} \\ C_{n_p} & C_{n_r} \end{pmatrix} \begin{pmatrix} \sin\theta \\ -\cos\theta \cos\phi \end{pmatrix} \frac{\dot{\psi} b}{2V}$$

where

$$\phi = \arctan\left(\frac{V}{g} \dot{\psi} \right).$$

Thus, for a given speed V, turn rate $\dot{\psi}$, and climb angle θ, one may compute the sideslip angle and the aileron and rudder deflections necessary to maintain a banked turn.

References

[1] B. Etkin and L. D. Reid. *Dynamics of Flight: Stability and Control.* John Wiley and Sons, New York, NY, third edition, 1996.