Lecture 6: Longitudinal Maneuvering Flight

Symmetric pull-up. Consider an aircraft in wings-level flight which is executing a steady pull-up, that is, a pitch-up maneuver at constant pitch rate. As a simple, but representative case, we will consider the situation where the airplane just passes through horizontal flight. The centripetal acceleration at the instant shown in Figure 1 is $\boldsymbol{a} = a_z \boldsymbol{k}$ where

$$a_z = -\frac{V^2}{R_{\text{pull-up}}}.$$

Now, if the airplane continued around the curved path and performed a circular loop at constant speed, it would have performed one full pitch rotation. The rate of pitch rotation is

$$q = \frac{V}{R_{\text{pull-up}}}.$$

Therefore,

$$a_z = -qV.$$

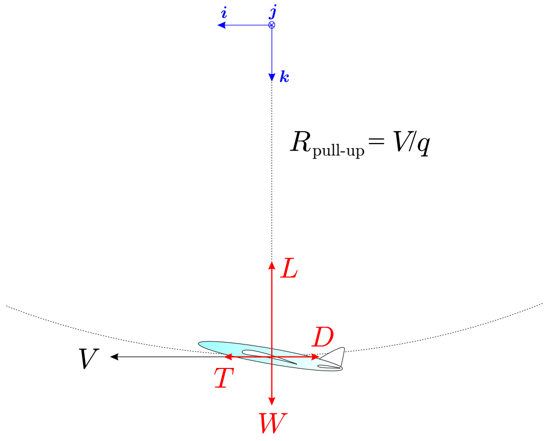

Figure 1: A wings-level pull-up.

Applying Newton's second law of motion gives

$$\begin{aligned} \Sigma F_x &= 0 & \Rightarrow \quad & T - D = 0 \\ \Sigma F_z &= m a_z & \Rightarrow \quad & W - L = -mqV \end{aligned}$$

Because we are considering steady motion (i.e., flight at constant $\boldsymbol{v}$ and $\boldsymbol{\omega}$), the aerodynamic moment must be zero.

Define the *load factor*

$$n := \frac{L}{W}.$$

In wings-level, equilibrium flight the load factor is one. The corresponding *normal acceleration* $(n-1)g$ is zero because the pilot is not accelerating. For a pull-up maneuver, the load factor increases. If $n = 2$, for example, the lift being generated is twice the weight of the aircraft, causing the aircraft to accelerate upward at *one* factor of g or, more briefly, "one g."

Dividing through by $W = mg$ in the z-direction force balance equation gives

$$1 - n = -\frac{1}{g} qV.$$

Solving for pitch rate in terms of velocity and load factor, we find for a symmetric pull-up:

$$\boxed{q = (n-1)\frac{g}{V}.}$$

Effect of pitch rate on lift force and pitch moment. We would like to know how the two maneuvering flight conditions discussed above relate to the elevator control authority. In particular, we would like to have some measure of the elevator angle necessary to execute a given maneuver, that is, the *elevator angle per g* for this maneuvering flight condition. To do this, we must first account for the effect of pitch rate on the lift force and pitch moment.

Previously, we considered only equilibrium flight. We therefore assumed that lift and pitch moment depended only on α and δe. More generally, in longitudinal maneuvering flight, lift and pitch moment also vary with pitch rate. We assume that the dependence is linear:

$$\begin{aligned} C_m &= C_{m_{0\mathrm{L}}} + C_{m_\alpha}\alpha + C_{m_{\delta e}}\delta e + \frac{\partial C_m}{\partial q}q \\ C_L &= C_{L_\alpha}\alpha + C_{L_{\delta e}}\delta e + \frac{\partial C_L}{\partial q}q. \end{aligned}$$

Notice that every stability derivative introduced thus far has been dimensionless. (Consider, for example, C_{m_α} or $C_{m_{\delta e}}$.) This is because angles, measured in radians, are dimensionless. On the other hand, angular rate has units of radians per second. To keep things nondimensional, we define a *dimensionless pitch rate*[1]

$$\hat{q} := \frac{q}{\frac{V}{\bar{c}/2}} = \frac{\bar{c}}{2V}q.$$

(Note the enigmatic factor of $\frac{1}{2}$ which is an artifact of early literature on aeroelasticity.) We now define the stability derivatives

$$\boxed{\begin{aligned} C_{L_q} &= \frac{\partial C_L}{\partial \hat{q}} = \frac{2V}{\bar{c}}\frac{\partial C_L}{\partial q} \\ C_{m_q} &= \frac{\partial C_m}{\partial \hat{q}} = \frac{2V}{\bar{c}}\frac{\partial C_m}{\partial q}. \end{aligned}}$$

To determine the value of C_{m_q} we consider the incremental contribution of pitch rate to pitch moment due to an increase in lift generated by the tail. This increase in lift is the effect of an increased angle of attack at the tail due to pitch rotation about the center of gravity. Of course, the wing experiences a similar change in angle of attack, but the moment arm from the wing aerodynamic center to the aircraft CG is small compared with l_t and the contribution to the total pitch moment is generally small. Typically, one accounts for the small contribution to C_{m_q} from the wing and fuselage by including a correction factor in the contribution due to the tail.

Consider the sketch in Figure 2. The *increment* in tail angle of attack is

$$\Delta\alpha_t = \frac{ql_t}{V}$$

The increment in the *tail* lift coefficient due to pitch rate is

$$\Delta C_{L_t} = C_{L_{\alpha_t}}\Delta\alpha_t = \left(C_{L_{\alpha_t}}\frac{ql_t}{V}\right)$$

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[1] See Table 4.1 on page 116 of [1] for a complete list of definitions for nondimensional forms.

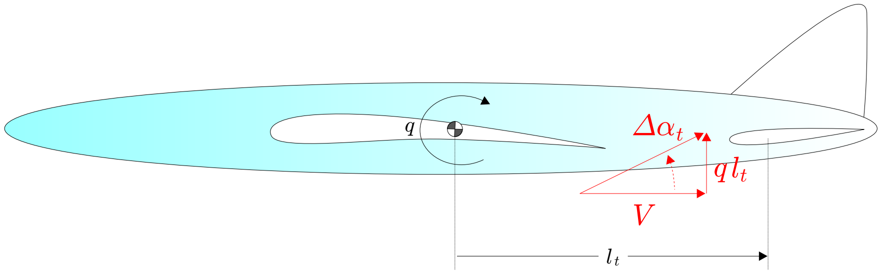

Figure 2: Effect of pitch rate on tail angle of attack.

The increment in *total* lift coefficient is

$$\begin{aligned}
\Delta C_L &= \frac{1}{\left(\frac{1}{2}\rho V^2 S\right)} L_t \\
&= \frac{1}{\left(\frac{1}{2}\rho V^2 S\right)} \left\{ \Delta C_{L_t} \left(\frac{1}{2} \rho V^2 S_t \right) \right\} \\
&= C_{L_{\alpha_t}} \frac{q l_t}{V} \frac{S_t}{S}.
\end{aligned}$$

Differentiating with respect to q and nondimensionalizing gives

$$\begin{aligned}
C_{L_q} &= \frac{2V}{\bar{c}} \left(\frac{C_{L_{\alpha_t}} l_t}{V} \right) \frac{S_t}{S} \\
&= 2 C_{L_{\alpha_t}} \left(\frac{S_t l_t}{S \bar{c}} \right)
\end{aligned}$$

or

$$\boxed{C_{L_q} = 2 C_{L_{\alpha_t}} \not{V}_{\mathrm{H}}.}$$

Turning now to the pitch moment coefficient, we note that the pitch moment due to the increment in tail lift is

$$\Delta M_t = -l_t \Delta C_{L_t} \left(\frac{1}{2} \rho V^2 S_t \right).$$

Nondimensionalizing gives

$$\begin{aligned}
\Delta C_{m_t} &= -\frac{l_t S_t}{\bar{c} S} \Delta C_{L_t} \\
&= -\not{V}_{\mathrm{H}} \left(C_{L_{\alpha_t}} \frac{q l_t}{V} \right).
\end{aligned}$$

Differentiating with respect to q and nondimensionalizing once again gives

$$\begin{aligned}
C_{m_{t_q}} &= -\frac{2V}{\bar{c}} \left(\frac{C_{L_{\alpha_t}} l_t}{V} \right) \not{V}_{\mathrm{H}} \\
&= -2 C_{L_{\alpha_t}} \not{V}_{\mathrm{H}} \frac{l_t}{\bar{c}}
\end{aligned}$$

A factor $k \approx 1.1$ is generally applied to account for wing-fuselage effects. Thus, we have

$$\boxed{C_{m_q} = -2kC_{L_{\alpha_t}} \mathcal{V}_{\text{H}} \frac{l_t}{\bar{c}}.}$$

Elevator increment and elevator angle per g. Recall that, for equilibrium flight, the trim conditions were

$$\begin{aligned} C_{L_{\text{trim}}} = C_W &= C_{L_\alpha}\alpha_{\text{trim}} + C_{L_{\delta e}}\delta e_{\text{trim}} && (1) \\ C_{m_{\text{trim}}} = 0 \quad &= C_{m_0} + C_{m_\alpha}\alpha_{\text{trim}} + C_{m_{\delta e}}\delta e_{\text{trim}}. && (2) \end{aligned}$$

Solving these two linear algebraic equations in two unknowns gives the angle of attack and elevator deflection for equilibrium flight. Now, however, we are considering the case of maneuvering flight. For both of the cases considered (a pull-up and a horizontal turn), the angular rate is constant. The equations above become

$$\begin{aligned} C_{L_{\text{maneuver}}} = nC_W &= C_{L_\alpha}\alpha_{\text{maneuver}} + C_{L_{\delta e}}\delta e_{\text{maneuver}} + C_{L_q}\hat{q}_{\text{maneuver}} && (3) \\ C_{m_{\text{maneuver}}} = 0 \qquad &= C_{m_0} + C_{m_\alpha}\alpha_{\text{maneuver}} + C_{m_{\delta e}}\delta e_{\text{maneuver}} + C_{m_q}\hat{q}_{\text{maneuver}}. && (4) \end{aligned}$$

Subtracting equations (1) and (2) for wings-level, equilibrium flight from equations (3) and (4) for maneuvering flight leaves

$$\begin{aligned} (n-1)C_W &= C_{L_\alpha}\Delta\alpha + C_{L_{\delta e}}\Delta\delta e + C_{L_q}\hat{q} \\ 0 &= C_{m_\alpha}\Delta\alpha + C_{m_{\delta e}}\Delta\delta e + C_{m_q}\hat{q} \end{aligned}$$

where

$$\Delta\alpha = \alpha_{\text{maneuver}} - \alpha_{\text{trim}} \qquad \text{and} \qquad \Delta\delta e = \delta e_{\text{maneuver}} - \delta e_{\text{trim}}.$$

Rearranging the equations above, we have

$$\begin{pmatrix} C_{L_\alpha} & C_{L_{\delta e}} \\ C_{m_\alpha} & C_{m_{\delta e}} \end{pmatrix} \begin{pmatrix} \Delta\alpha \\ \Delta\delta e \end{pmatrix} = \begin{pmatrix} (n-1)C_W - C_{L_q}\hat{q} \\ -C_{m_q}\hat{q} \end{pmatrix}.$$

The solution to these equations is

$$\begin{pmatrix} \Delta\alpha \\ \Delta\delta e \end{pmatrix} = \frac{1}{\text{Det}} \begin{pmatrix} C_{m_{\delta e}} & -C_{L_{\delta e}} \\ -C_{m_\alpha} & C_{L_\alpha} \end{pmatrix} \begin{pmatrix} (n-1)C_W - C_{L_q}\hat{q} \\ -C_{m_q}\hat{q} \end{pmatrix}$$

where

$$\text{Det} = C_{L_\alpha}C_{m_{\delta e}} - C_{L_{\delta e}}C_{m_\alpha},$$

as before. (Recall that Det is normally negative.) The increment in elevator deflection for a given maneuver is

$$\Delta\delta e = \frac{1}{\text{Det}}\left[-C_{m_\alpha}\left((n-1)C_W - C_{L_q}\hat{q}\right) + C_{L_\alpha}\left(-C_{m_q}\hat{q}\right)\right]$$

or

$$\Delta\delta e = \frac{1}{\text{Det}}\left[-C_{m_\alpha}(n-1)C_W + \left(C_{m_\alpha}C_{L_q} - C_{L_\alpha}C_{m_q}\right)\hat{q}\right] \tag{5}$$

Notice that $\Delta\delta e$ vanishes when $n = 1$ and $\hat{q} = 0$, as it must.

Symmetric pull-ups. Recall that, for a pull-up maneuver,

$$q = (n-1)\frac{g}{V}.$$

Thus, we have

$$
\begin{aligned}
\hat{q} = \frac{q\bar{c}}{2V} &= (n-1)\frac{g\bar{c}}{2V^2} \\
&= (n-1)\frac{g\bar{c}}{2V^2} \cdot \frac{m\rho S}{m\rho S} \\
&= (n-1)\left(\frac{mg}{\frac{1}{2}\rho V^2 S}\right)\frac{\rho\bar{c}S}{4m} \\
&= (n-1)C_W\frac{\rho\bar{c}S}{4m}.
\end{aligned}
$$

Substituting into the equation for $\Delta\delta e$ gives the *elevator increment for a pull-up maneuver*

$$
\boxed{\Delta\delta e = \frac{(n-1)C_W}{\text{Det}}\left[-C_{m_\alpha} + \left(C_{m_\alpha}C_{L_q} - C_{L_\alpha}C_{m_q}\right)\left(\frac{\rho\bar{c}S}{4m}\right)\right].}
$$

Dividing by the normal acceleration $(n-1)$ (as measured in "g's") gives the *elevator angle per g for a pull-up maneuver*:

$$
\boxed{\frac{\Delta\delta e}{n-1} = \frac{C_W}{\text{Det}}\left[-C_{m_\alpha} + \left(C_{m_\alpha}C_{L_q} - C_{L_\alpha}C_{m_q}\right)\left(\frac{\rho\bar{c}S}{4m}\right)\right].}
$$

Given the relationship between elevator hinge moment and stick force, one may use the expression above to compute the stick force per g, a critical parameter in design of the pilot interface.

In [1], the authors define the *stick-fixed maneuver point* h_m as that CG location at which the elevator angle per g (for a pull-up maneuver) is zero. Computing this number, one finds that the stick-fixed maneuver point is aft of the stick-fixed neutral point h_n, meaning that, for a statically stable aircraft, some nonzero elevator angle will be required to execute a pull-up maneuver. In general, the elevator angle per g will be proportional to $h_m - h$, which is referred to as the *stick-fixed maneuver margin.* The larger the maneuver margin, the larger the elevator deflection that is required to pull up at a given acceleration.

Horizontal turn at constant radius. Now consider an aircraft undergoing a steady turn at constant radius and constant altitude. Strictly speaking, this is not a purely longitudinal flight condition like the symmetric pull-up. In fact, the process of switching from wings-level equilibrium flight to a horizontal turn is fairly complicated, involving motion in all six degrees of freedom. However, the problem of *maintaining* a steady horizontal turn once it has been achieved is largely a question of longitudinal control.

As shown in Figure 3, the lift vector in a steady turn is deflected through the roll angle ϕ. The centripetal acceleration at the instant shown in Figure 3 is $\boldsymbol{a} = a_y\boldsymbol{j}$ where

$$
a_y = \frac{V^2}{R_{\text{turn}}} = \left(\frac{V}{R_{\text{turn}}}\right)V = \dot{\psi}V.
$$

Recalling that the vertical component of lift must balance weight for constant altitude flight, Newton's second law of motion gives

$$
\begin{aligned}
\Sigma F_x = 0 \qquad &\Rightarrow \qquad T - D = 0 \\
\Sigma F_y = ma_y \qquad &\Rightarrow \qquad L\sin\phi = m\dot{\psi}V \\
\Sigma F_z = 0 \qquad &\Rightarrow \qquad W - L\cos\phi = 0
\end{aligned}
$$

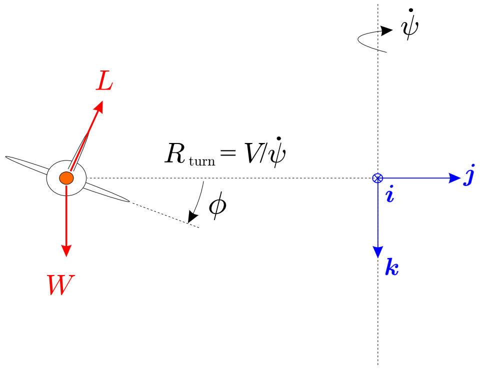

Figure 3: A steady turn at constant rate and altitude.

The general expression for pitch rate is

$$q = \dot\theta \cos\phi + \dot\psi \cos\theta \sin\phi$$

For symmetric motions, such as a pull-up, $\phi = 0$ and we find that $q = \dot\theta$. This is not the case for a steady turn. For a steady turn, the pitch angle θ remains constant; that is, $\dot\theta = 0$. The *pitch rate* q, however is nonzero. Assuming that the (constant) pitch angle is relatively small,

$$q \approx \dot\psi \sin\phi.$$

Replacing $\dot\psi$ in the y-direction force balance equation gives

$$L \sin\phi = \frac{mVq}{\sin\phi}.$$

Dividing by $W = mg$, substituting the load factor $n = \frac{L}{W}$ and rearranging gives

$$n \sin^2\phi = \frac{1}{g} V q.$$

Solving for pitch rate then gives

$$q = \frac{g}{V} n \sin^2\phi.$$

Now, from the z-direction force balance equation,

$$\cos\phi = \frac{1}{n} \qquad \Rightarrow \qquad \sin^2\phi = 1 - \cos^2\phi = 1 - \frac{1}{n^2}.$$

Thus, for a horizontal turn at constant altitude and radius, we have

$$\boxed{q = \left(n - \frac{1}{n}\right) \frac{g}{V}}$$

The nondimensional pitch rate is

$$\begin{aligned} \hat q = \frac{q\bar c}{2V} &= \left(n - \frac{1}{n}\right) \frac{g\bar c}{2V^2} \\ &= \left(n - \frac{1}{n}\right) C_W \frac{\rho \bar c S}{4m}. \end{aligned}$$

Substituting into equation (5) for $\Delta\delta e$ gives the *elevator angle required for a horizontal turn*:

$$\boxed{\Delta\delta e = (n-1)\frac{C_W}{\text{Det}}\left[-C_{m_\alpha} + \left(C_{m_\alpha}C_{L_q} - C_{L_\alpha}C_{m_q}\right)\left(\frac{n+1}{n}\right)\left(\frac{\rho\bar{c}S}{4m}\right)\right].}$$

The *elevator angle per g for a horizontal turn* is:

$$\boxed{\frac{\Delta\delta e}{n-1} = \frac{C_W}{\text{Det}}\left[-C_{m_\alpha} + \left(C_{m_\alpha}C_{L_q} - C_{L_\alpha}C_{m_q}\right)\left(\frac{n+1}{n}\right)\left(\frac{\rho\bar{c}S}{4m}\right)\right].}$$

Alternatively, given the necessary aircraft parameters and actuator limits, one could use the prior equation to determine the turn of smallest radius (or largest load factor) which a given aircraft can execute.

References

[1] B. Etkin and L. D. Reid. *Dynamics of Flight: Stability and Control.* John Wiley and Sons, New York, NY, third edition, 1996.