Lecture 5: Longitudinal Control

We next turn our attention to control of longitudinal motion, particularly control of the pitch attitude θ. For wings-level, equilibrium flight at constant altitude, α is equal to the pitch angle θ. Thus, we are interested in controlling α, and hence the lift generated by the aircraft.

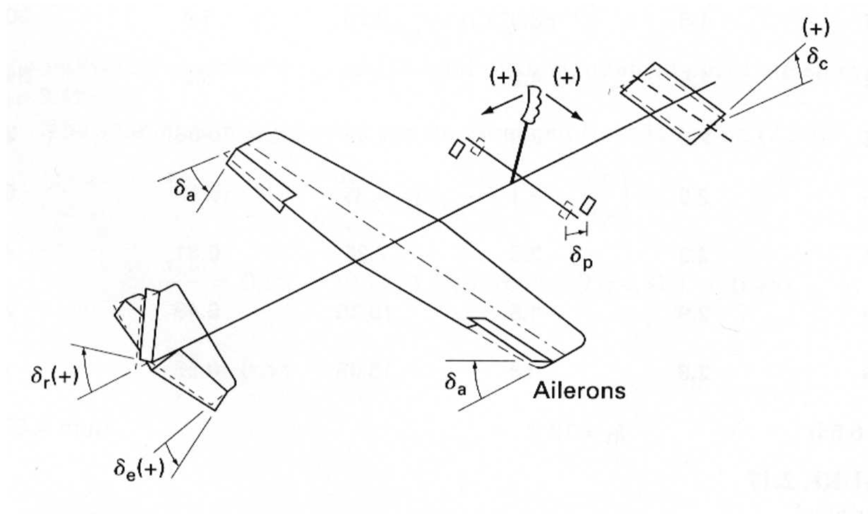

Figure 1: Standards for stick and control surface deflections [2].

For conventional aircraft, there are three primary control surfaces which provide moments about the three coordinate axes. Figure 1 depicts the elevator, rudder, and ailerons, as well as a pair of canards, which are somewhat less common. The figure also depicts the sign conventions for the various stick, pedal, and actuator deflections.

Aerodynamic control surfaces are generally small lifting surfaces which generate small local forces. Each of these forces acts through a large moment arm to generate a control moment about the aircraft center of gravity. Table 1 shows the effect of positive control surface deflections on the aerodynamic moments. For example, a positive elevator deflection (as indicated in Figure 1) results in a negative moment about the lateral axis: $M_{\delta e} < 0$, as we shall verify shortly. Because these control surfaces are simply lifting surfaces, everything we know about wings can be applied to our study of actuator effects.

Table 1: Effect of control surface deflections on the aerodynamic moments.

Actuator Name	Angle to be Controlled	Symbol for Actuator Deflection	Moment Sensitivity to Actuator Deflection
Aileron	ϕ	δa	$L_{\delta a} > 0$
Elevator	α (or θ)	δe	$M_{\delta e} < 0$
Rudder	β (or ψ)	δr	$N_{\delta r} < 0$

Primary concerns in designing a control surface, such as an elevator, are control effectiveness and hinge moments and stick forces. Control effectiveness relates to the ability of a given control surface to generate the necessary control moments. Hinge moments and stick forces relate to the pilot's ability to deflect the control surface as necessary.

We previously assumed that the lift of the aircraft was a function only of its angle of attack α. Now, we recognize that deflections δe of the elevator will also affect the lift. Assuming these deflections are small

($\delta e \ll 1$ radian), we may write (for the entire aircraft)

$$C_L = C_{L_\alpha}\alpha + C_{L_{\delta e}}\delta e.$$

The coefficient

$$C_{L_{\delta e}} = \frac{\partial C_L}{\partial \delta e}$$

is typically positive, but small relative to C_{L_α}. Thus, the effect of positive (downward) elevator deflections is a small increase in the value of C_L.

The more important effect of the elevator is on the pitch moment. The aircraft moment coefficient is

$$C_m = C_{m_0} + C_{m_\alpha}\alpha + C_{m_{\delta e}}\delta e$$

where

$$C_{m_{\delta e}} = \frac{\partial C_m}{\partial \delta e}$$

is referred to as the *elevator control power* or simply elevator power. The elevator power determines how effective the elevator is at generating pitch control moments.

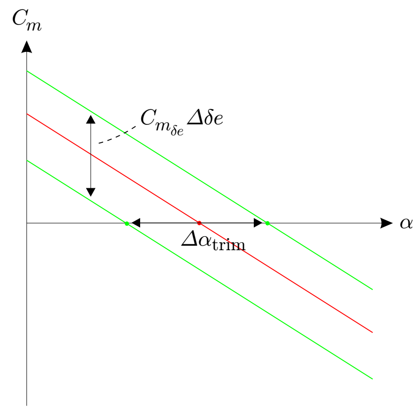

Figure 2: Effect of elevator deflections on C_m and α_{trim}.

As shown in Figure 2, the effect of elevator deflections is to shift the C_m versus α curve up or down. (Which direction the curve moves, depends on the sign of $C_{m_{\delta e}}$.) Equivalently, the effect is to shift the trim angle of attack α_{trim} right or left. Intuition suggests that negative elevator displacements, which correspond to positive (backward) deflections of the stick, result in higher trim angles of attack. We would therefore expect that $C_{m_{\delta e}} < 0$ so that negative (upward) elevator deflections correspond to positive (nose-up) pitch moments.

To verify this intuition about the elevator power, we examine the effect of the elevator more carefully. The increase in lift provided by a positive elevator deflection is

$$\Delta L = \Delta C_L \left(\frac{1}{2}\rho V^2\right) S.$$

At a *fixed* angle of attack, this increase is generated entirely by the horizontal tail, so

$$\Delta L = \Delta L_t = \Delta C_{L_t} \left(\frac{1}{2}\rho V^2\right) S_t.$$

We therefore find that

$$\begin{aligned} \Delta C_L &= \frac{S_t}{S}\Delta C_{L_t} \\ &= \frac{S_t}{S}\left(\frac{\partial C_{L_t}}{\partial \delta e}\delta e\right). \end{aligned}$$

The coefficient

$$a_e = \frac{\partial C_{L_t}}{\partial \delta e}$$

is termed the *elevator effectiveness* and is assumed to be constant for a given horizontal tail and elevator assembly. The elevator effectiveness determines how effective the elevator is at modifying the lift generated by the tail.[1]

If the elevator is a flap at the trailing edge of the horizontal stabilizer, as is typically the case, one may write

$$a_e = \frac{\partial C_{L_t}}{\partial \alpha_t} \frac{d\alpha_t}{d\delta e}$$

or

$$\boxed{a_e = \frac{\partial C_{L_t}}{\partial \alpha_t} \tau}$$

where the constant

$$\tau = \frac{d\alpha_t}{d\delta e}$$

depends on the ratio of the elevator surface area to the complete tail lifting surface area. If this ratio of areas is one, i.e., if the entire tail can be deflected as an elevator, then $\tau = 1$. The parameter τ can be determined as described in Appendix B.2 of [1].

Differentiating the expression for ΔC_L given above with respect to the elevator deflection δe gives

$$\boxed{C_{L_{\delta e}} = a_e \frac{S_t}{S}}$$

Turning now to the primary purpose for the elevator, as a pitch moment generator, we see that the increment in total pitch moment due to an elevator deflection is

$$\Delta M \approx -l_t \Delta L_t = -l_t \left(\Delta C_{L_t} \left(\frac{1}{2} \rho V^2 \right) S_t \right)$$

Here, as in previous discussions, we have neglected the small contributions due to the change in drag on the tail and the height of the tail relative to the fuselage. Normalizing by $\left(\frac{1}{2}\rho V^2\right) S\bar{c}$ gives

$$\Delta C_m = -a_e \mathcal{V}_{\text{H}} \delta e.$$

Since we assume the angle of attack remains fixed, $\Delta C_m = C_{m_{\delta e}} \delta e$ and the elevator power is simply

$$\boxed{C_{m_{\delta e}} = -a_e \mathcal{V}_{\text{H}}}$$

Note that $C_{m_{\delta e}}$ is indeed negative for an aft tail. Thus, a negative elevator deflection (trailing edge upward) yields a positive (nose-up) pitching moment.

Longitudinal trim conditions: $\boldsymbol{\delta e \neq 0}$. For wings-level, equilibrium flight, we need

[1] Note: "Elevator effectiveness" is the tail lift coefficient sensitivity to elevator deflections while "elevator power" is the moment coefficient sensitivity to elevator deflections.

- the thrust to balance the drag (which we assume is the case)
- the lift to balance the weight:

$$C_{L_{\text{trim}}}\left(\frac{1}{2}\rho V^2\right)S = W$$

  where

$$C_{L_{\text{trim}}} = C_{L_\alpha}\alpha_{\text{trim}} + C_{L_{\delta e}}\delta e_{\text{trim}},$$

- the pitching moment to vanish

$$0 = C_{m_0} + C_{m_\alpha}\alpha_{\text{trim}} + C_{m_{\delta e}}\delta e_{\text{trim}}.$$

For an airplane of given geometry, the aerodynamic coefficients and stability derivatives are known. The only unknowns are the angle of attack and elevator deflection. Rearranging these two linear algebraic equations into matrix form, we have

$$\begin{pmatrix} C_{L_\alpha} & C_{L_{\delta e}} \\ C_{m_\alpha} & C_{m_{\delta e}} \end{pmatrix}\begin{pmatrix} \alpha_{\text{trim}} \\ \delta e_{\text{trim}} \end{pmatrix} = \begin{pmatrix} C_{L_{\text{trim}}} \\ -C_{m_0} \end{pmatrix}. \tag{1}$$

The solution to these equations is

$$\begin{aligned}
\begin{pmatrix} \alpha_{\text{trim}} \\ \delta e_{\text{trim}} \end{pmatrix} &= \begin{pmatrix} C_{L_\alpha} & C_{L_{\delta e}} \\ C_{m_\alpha} & C_{m_{\delta e}} \end{pmatrix}^{-1}\begin{pmatrix} C_{L_{\text{trim}}} \\ -C_{m_0} \end{pmatrix} \\
&= \frac{1}{\text{Det}}\begin{pmatrix} C_{m_{\delta e}} & -C_{L_{\delta e}} \\ -C_{m_\alpha} & C_{L_\alpha} \end{pmatrix}\begin{pmatrix} C_{L_{\text{trim}}} \\ -C_{m_0} \end{pmatrix} \\
&= \frac{1}{\text{Det}}\begin{pmatrix} C_{m_{\delta e}}C_{L_{\text{trim}}} + C_{L_{\delta e}}C_{m_0} \\ -C_{m_\alpha}C_{L_{\text{trim}}} - C_{L_\alpha}C_{m_0} \end{pmatrix}
\end{aligned}$$

where "Det" represents the determinant of the matrix on the left hand side of (1):

$$\text{Det} = C_{L_\alpha}C_{m_{\delta e}} - C_{L_{\delta e}}C_{m_\alpha}.$$

The first term $C_{L_\alpha}C_{m_{\delta e}}$ is generally negative, as we have seen, while the second term $-C_{L_{\delta e}}C_{m_\alpha}$ is positive. Because the first term typically dominates, however, the determinant Det is typically negative.

Maximum forward CG location. The pitch moment coefficient for the aircraft is

$$C_m = C_{m_0} + C_{m_\alpha}\alpha + C_{m_{\delta e}}\delta e.$$

The purpose of the elevator is to provide a counter-moment to $C_{m_0} + C_{m_\alpha}\alpha$ so that the aircraft can be trimmed at different angles of attack. Recall that, as the CG moves forward, the C_m curve becomes steeper. This effect enhances stability of wings-level equilibrium flight, however it simultaneously reduces the effectiveness of the elevator because greater and greater trim deflections are needed to obtain a given change in the trim angle of attack; see Figure 3.

For an aircraft with given physical limits on the elevator deflection angle, we may compute the maximum forward CG location to be that CG location for which the minimum elevator deflection corresponds to stall. That is, deflecting the elevator such that its trailing edge is at its *upper* limit should bring the aircraft to the brink of stall:

$$\delta e_{\text{trim}} = \delta e_{\text{min}} \qquad \Rightarrow \qquad C_{L_{\text{trim}}} = C_{L_{\text{max}}}.$$

If the CG were any further forward than this point, the aircraft would be overly stable in pitch; the range of achievable equilibrium states would be restricted by the actuator limits rather than by the flow physics. This could be problematic in landing, for example, where a low airspeed requires a large lift coefficient.

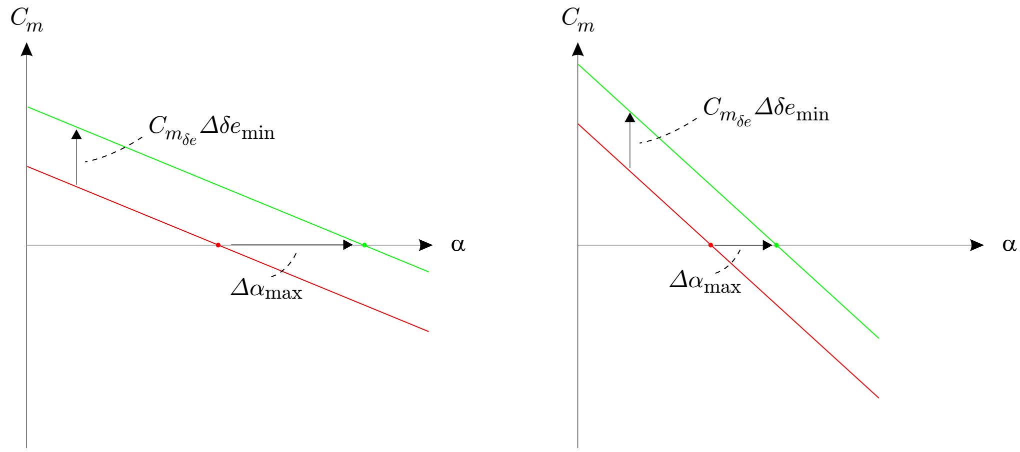

Figure 3: Illustration that elevator is less effective when C_{m_α} is more negative.

From the equation for the trim elevator deflection, we require

$$(C_{L_\alpha} C_{m_{\delta e}} - C_{L_{\delta e}} C_{m_\alpha})\delta e_{\min} = -C_{m_\alpha} C_{L_{\max}} - C_{L_\alpha} C_{m0} \tag{2}$$

where $C_{L_{\max}}$ is the lift coefficient at stall. In this equation, the only terms which depend on the CG location are $C_{m_{\delta e}}$ and C_{m_α}:

$$\begin{aligned} C_{m_\alpha} &= C_{L_\alpha}(h - h_n) \\ C_{m_{\delta e}} &= -a_e \mathcal{V}_{\mathrm{H}} \end{aligned}$$

Assuming that the elevator is well aft of the aircraft CG, the horizontal tail volume remains relatively constant, so only C_{m_α} depends significantly on the CG location. Rearranging terms in (2), we obtain

$$(C_{m_0} + C_{m_{\delta e}} \delta e_{\min})\, C_{L_\alpha} = -C_{m_\alpha}(C_{L_{\max}} - C_{L_{\delta e}} \delta e_{\min})$$

Substituting

$$C_{m_\alpha} = C_{L_\alpha}(h - h_n)$$

and solving for forward-most acceptable CG location gives

$$\boxed{h_{\min} = h_n - \frac{C_{m_0} + C_{m_{\delta e}} \delta e_{\min}}{C_{L_{\max}} - C_{L_{\delta e}} \delta e_{\min}}.}$$

By definition, $h_{\min}$ is the (nondimensional) CG location at which the trim elevator deflection $\delta e_{\min}$ yields the maximum lift coefficient $C_{L_{\max}}$. If the CG lies aft of $h_{\min}$, then the full range of possible lift coefficients will be obtainable by the pilot.

References

[1] B. Etkin and L. D. Reid. *Dynamics of Flight: Stability and Control*. John Wiley and Sons, New York, NY, third edition, 1996.

[2] R. C. Nelson. *Flight Stability and Automatic Control*. WCB McGraw-Hill, New York, NY, second edition, 1998.