# Lecture 4: Total Pitch Moment Coefficient

Recall that, in Lecture 3, we found

$$C_{m_t} = -\mathcal{V}_{\rm H} C_{L_{\alpha_t}} \alpha_t$$

where $\mathcal{V}_{\rm H} = \frac{S_t l_t}{S\bar{c}}$ is the horizontal tail volume ratio (with $l_t$ representing the signed longitudinal distance from the center of gravity to the tail aerodynamic center). By definition,

$$\alpha_t = \alpha_{wb} - \epsilon(\alpha_{wb}) - i_t,$$

where the downwash angle $\epsilon(\alpha_{wb})$ depends on the lift being generated by the wing and body. For reasonable angles of attack, one may approximate $\epsilon(\alpha_{wb})$ by the first two terms of its Taylor series expansion about $\alpha_{wb} = 0$. Assuming zero wing incidence ($i_w = 0$), as Etkin and Reid do,

$$\epsilon = \epsilon(0) + \left.\frac{\partial \epsilon}{\partial \alpha_{wb}}\right|_{\alpha_{wb}=0} \alpha_{wb} \tag{1}$$

To simplify notation, and to remain consistent with the textbook, we will write

$$\epsilon_0 = \epsilon(0) \qquad \text{and} \qquad \frac{\partial \epsilon}{\partial \alpha} = \left.\frac{\partial \epsilon}{\partial \alpha_{wb}}\right|_{\alpha_{wb}=0}.$$

The coefficients $\epsilon_0$ and $\frac{\partial \epsilon}{\partial \alpha}$ may be estimated as described in Appendix B of [1]. Note that, because of wing twist and other effects, $\epsilon_0$ may be nonzero even if the wing/body generates no net lift when $\alpha_{wb} = 0$.

Substituting for $\epsilon$ from (1) in the definition of $\alpha_t$, we find that

$$\alpha_t = \left(1 - \frac{\partial \epsilon}{\partial \alpha}\right) \alpha_{wb} - \epsilon_0 - i_t. \tag{2}$$

The contribution to $C_m$ from the horizontal tail may therefore be written

$$\boxed{\begin{aligned} C_{m_t} &= -\mathcal{V}_{\rm H} C_{L_{\alpha_t}} \alpha_t \\ &= C_{m_{0_t}} + C_{m_{\alpha_t}} \alpha_{wb} \\ &\text{where} \\ C_{m_{0_t}} &= \mathcal{V}_{\rm H} C_{L_{\alpha_t}} (\epsilon_0 + i_t) \\ C_{m_{\alpha_t}} &= -\mathcal{V}_{\rm H} C_{L_{\alpha_t}} \left(1 - \frac{\partial \epsilon}{\partial \alpha}\right). \end{aligned}}$$

Consider the term $C_{m_{0_t}}$. Recall that we require a positive pitch moment at zero lift so that the aircraft can be balanced at a positive angle of attack (i.e., so that the aircraft can generate appropriate lift in equilibrium flight). Also recall that the wing's contribution to the zero-lift pitch moment is negative for airfoils with positive camber (the most commonly used type). An aft tail's contribution to $C_{m_0}$ will be positive provided

$$i_t > -\epsilon_0.$$

By making $i_t$ large enough, one may increase the positive contribution to $C_{m_0}$. (Recall that $i_t$ is positive when the leading edge of the horizontal tail is rotated *downward*.) The effectiveness of the horizontal tail's contribution may be increased by increasing the moment arm $l_t$ or by increasing the tail surface area $S_t$.

Considering next the term $C_{m_{\alpha_t}}$, and noting that typically $0 < \frac{\partial \epsilon}{\partial \alpha} < 1$, it is clear that an aft tail will provide a stabilizing (negative) increment in $C_{m_\alpha}$, as well. This increment can be increased by increasing the horizontal tail volume ratio and also by increasing the lift curve slope $C_{L_{\alpha_t}}$.

Similar analysis can be performed for a forward tail, or canard, with a few notable differences. First, the local airflow forward of the wing experiences an "upwash" rather than a downwash; this tends to increase the local angle of attack seen at the canard. Second, the signed horizontal distance $l_t$ from the CG to the canard will likely be negative. As a result, a canard's contribution to $C_{m_\alpha}$ reduces the airplane's static longitudinal stability.

**Propulsion effects.** Formally, we may approximate the propulsion system's contribution to pitch moment as

$$C_{m_p} = C_{m_{0_p}} + C_{m_{\alpha_p}} \alpha_{wb},$$

however the task of determining the coefficients $C_{m_{0_p}}$ and $C_{m_{\alpha_p}}$ can be complicated. The simplest estimate involves taking the moment of the propulsion force about the CG, however there are other important effects as well. For example, a propeller or a jet engine at angle of attack experiences a force component normal to the thrust generated.

**Total pitching moment.** We may now write the total pitching moment coefficient for an aircraft in wings-level equilibrium flight:

$$C_m = C_{m_{wb}} + C_{m_t} + C_{m_p}.$$

(When the subscript "cg" is omitted, it should be assumed that the moment is taken about the aircraft center of gravity.) Written more explicitly,

$$C_m = \left( C_{m_{\mathrm{ac}_{wb}}} + C_{L_{\alpha_{wb}}} (h - h_{n_{wb}}) \alpha_{wb} \right) - \mathbb{V}_{\mathrm{H}} C_{L_{\alpha_t}} \alpha_t + C_{m_p}.$$

Substituting for $\alpha_t$ from (2), we may express the pitch moment coefficient as

$$C_m = \bar{C}_{m_0} + \bar{C}_{m_\alpha} \alpha_{wb}$$

where

$$\begin{aligned} \bar{C}_{m_0} &= C_{m_{\mathrm{ac}_{wb}}} + C_{m_{0_t}} + C_{m_{0_p}} \\ &= C_{m_{\mathrm{ac}_{wb}}} + \mathbb{V}_{\mathrm{H}} C_{L_{\alpha_t}} (\epsilon_0 + i_t) + C_{m_{0_p}} \end{aligned}$$

and

$$\begin{aligned} \bar{C}_{m_\alpha} &= C_{m_{\alpha_{wb}}} + C_{m_{\alpha_t}} + C_{m_{\alpha_p}} \\ &= C_{L_{\alpha_{wb}}} (h - h_{n_{wb}}) - \mathbb{V}_{\mathrm{H}} \left( 1 - \frac{\partial \epsilon}{\partial \alpha} \right) C_{L_{\alpha_t}} + C_{m_{\alpha_p}}. \end{aligned}$$

The zero-lift line for the entire aircraft is not the same, in general, as the zero-lift line for the wing and body. This is because the horizontal tail modifies the airplane's lift characteristics. Indeed, the entire point of the tail incidence angle $i_t$ is to generate lift to provide a nose-up moment when the wing and body are not creating sufficient lift to stay aloft. A side effect is a shift in the zero-lift line of the aircraft. Note that this moment-generating lift force generated by the tail is *downward* for an aft tail and *upward* for a canard. An advantage of a canard is that it not only provides a nose-up moment, but it also provides a lift force that contributes usefully to the overall lift generated by the aircraft. On the other hand, canards typically give a *positive* increment in $C_{m_\alpha}$ which tends to destabilize steady, wings level flight.

Defining a new angle of attack $\alpha$, measured to the zero-lift line of the entire aircraft, one obtains

$$C_L(\alpha) = C_{L_\alpha}\alpha. \tag{3}$$

Now, the lift generated by the entire aircraft is simply the sum of the lift generated by the wing and body and that generated by the tail:

$$L = L_{wb} + L_t.$$

Normalizing by $(\frac{1}{2}\rho V^2)S$, we find that

$$\begin{aligned} C_L &= C_{L_{wb}} + \frac{S_t}{S}C_{L_t} \\ &= C_{L_{\alpha_{wb}}}\alpha_{wb} + \frac{S_t}{S}C_{L_{\alpha_t}}\alpha_t \\ &= C_{L_{\alpha_{wb}}}\alpha_{wb} + \frac{S_t}{S}C_{L_{\alpha_t}}\left(\alpha_{wb}\left(1 - \frac{\partial\epsilon}{\partial\alpha}\right) - \epsilon_0 - i_t\right) \\ &= C_{L_{\alpha_{wb}}}\left(1 + \frac{S_t}{S}\frac{C_{L_{\alpha_t}}}{C_{L_{\alpha_{wb}}}}\left(1 - \frac{\partial\epsilon}{\partial\alpha}\right)\right)\alpha_{wb} - \frac{S_t}{S}C_{L_{\alpha_t}}\left(\epsilon_0 + i_t\right). \end{aligned}$$

(Recall that any effect due to a modified dynamic pressure at the tail is included in the tail lift coefficient.) Thus, we have

$$C_L(\alpha_{wb}) = C_{L_0} + C_{L_\alpha}\alpha_{wb} \tag{4}$$

where

$$C_{L_0} = -\frac{S_t}{S}C_{L_{\alpha_t}}\left(\epsilon_0 + i_t\right)$$

and

$$C_{L_\alpha} = C_{L_{\alpha_{wb}}}\left(1 + \frac{S_t}{S}\frac{C_{L_{\alpha_t}}}{C_{L_{\alpha_{wb}}}}\left(1 - \frac{\partial\epsilon}{\partial\alpha}\right)\right).$$

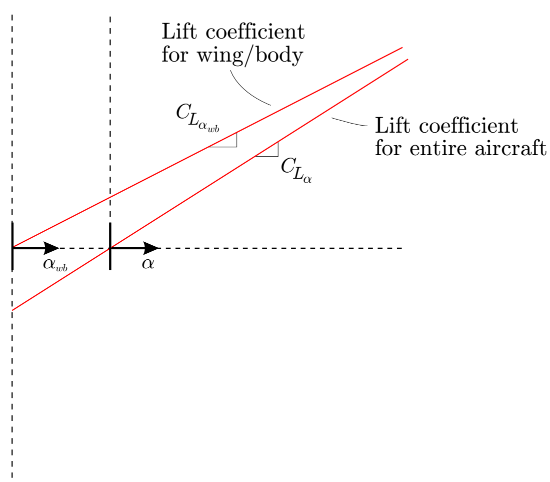


Figure 1: Lift coefficient curves for the wing/body and for the entire aircraft.

Comparing equations (3) and (4), we see that

$$\boxed{\alpha = \alpha_{wb} - \frac{S_t}{S}\frac{C_{L_{\alpha_t}}}{C_{L_\alpha}}\left(\epsilon_0 + i_t\right).}$$

Note that downwash and positive tail incidence (for an aft tail) *reduce* the effective angle of attack of the aircraft. This makes sense, of course – the lift generated by the tail due to these effects is opposite to the lift generated by the wing/body. Perhaps a better way describe the effect of an aft tail is that it shifts the wing/body lift coefficient curve downward but increases its slope; see Figure 1.

For completeness, we may restate our previous results concerning the total pitch moment in terms of the new angle of attack $\alpha$.

$$C_m = C_{m_0} + C_{m_\alpha}\alpha$$

where

$$C_{m_0} = C_{m_{\mathrm{ac}_{wb}}} + \underbrace{\left(\mathcal{V}_{\mathrm{H}} + \frac{S_t}{S}\left(h - h_{n_{wb}}\right)\right)}_{\bar{\mathcal{V}}_{\mathrm{H}}}\left(1 - \frac{S_t}{S}\frac{C_{L_{\alpha_t}}}{C_{L_\alpha}}\left(1 - \frac{\partial\epsilon}{\partial\alpha}\right)\right)C_{L_{\alpha_t}}\left(\epsilon_0 + i_t\right) + C_{m_{0_p}}$$

and

$$C_{m_\alpha} = C_{L_\alpha}\left(h - h_{n_{wb}}\right) - \underbrace{\left(\mathcal{V}_{\mathrm{H}} + \frac{S_t}{S}\left(h - h_{n_{wb}}\right)\right)}_{\bar{\mathcal{V}}_{\mathrm{H}}}\left(1 - \frac{\partial\epsilon}{\partial\alpha}\right)C_{L_{\alpha_t}} + C_{m_{\alpha_p}}.$$

The derivation is given in [1], where it is also noted that

$$\begin{aligned}
\bar{\mathcal{V}}_{\mathrm{H}} &= \mathcal{V}_{\mathrm{H}} + \frac{S_t}{S}\left(h - h_{n_{wb}}\right) \\
&= \frac{S_t l_t}{S\bar{c}} + \frac{S_t}{S}\left(h - h_{n_{wb}}\right) \\
&= \frac{S_t}{S}\left(h_t - h\right) + \frac{S_t}{S}\left(h - h_{n_{wb}}\right) \\
&= \frac{S_t}{S}\left(h_t - h_{n_{wb}}\right)
\end{aligned}$$

is invariant to changes in the aircraft CG location.

**The Neutral Point.** Because the location of the CG of an aircraft varies in flight as fuel is spent, passengers move about, etc, it is of interest to know the absolute limit on the range of CG motion for $C_{m_\alpha}$ to be negative.

**Definition.** The *neutral point* is the CG location $h = h_n$ at which $C_{m_\alpha} = 0$.

Essentially, the neutral point is the "aerodynamic center" for the complete airplane. Because, by definition, the slope of the $C_m$ curve is zero when $h = h_n$, the aircraft is *neutrally stable* in pitch when the CG is located at the neutral point. (Recall that $h$ denotes the location of the CG.) Thus, the trim condition will not be restored in response to a small deviation from the equilibrium pitch angle. For $h > h_n$, i.e., when the CG is aft of the neutral point, the aircraft will diverge from the equilibrium in response to a small pitch disturbance. Clearly $h = h_n$ is a critical condition which must be avoided.

To find $h_n$, we set $C_{m_\alpha}$ equal to zero and solve for $h = h_n$, which gives

$$h_n = h_{n_{wb}} + \bar{\mathcal{V}}_{\mathrm{H}}\left(1 - \frac{d\epsilon}{d\alpha}\right)\frac{C_{L_{\alpha_t}}}{C_{L_\alpha}} - \frac{C_{m_{\alpha_p}}}{C_{L_\alpha}}. \qquad (5)$$

The second term on the right hand side in (5) is generally positive for an aft tail, indicating that the tail shifts the neutral point aft, enhancing static stability.

It is common to replace the wing/body neutral point in the boxed expressions above with the aircraft neutral point $h_n$. Rearranging (5) to obtain

$$h_{n_{wb}} = h_n - \bar{V}_{\mathrm{H}} \left(1 - \frac{d\epsilon}{d\alpha}\right) \frac{C_{L_{\alpha_t}}}{C_{L_\alpha}} + \frac{C_{m_{\alpha_p}}}{C_{L_\alpha}}$$

and substituting wherever $h_{n_{wb}}$ appears gives

$$\boxed{\begin{gathered} C_m = C_{m_0} + C_{m_\alpha}\alpha \\ \text{where} \\ C_{m_0} = C_{m_{0L}} \quad \text{and} \quad C_{m_\alpha} = C_{L_\alpha}(h - h_n). \end{gathered}}$$

In the expression for $C_{m_0}$ above, $C_{m_{0L}}$ is the constant, zero-lift pitch moment coefficient for the *entire* airplane.

**Definition** The *static margin* for an aircraft is

$$K_n = h_n - h.$$

The static margin is a margin of safety for static longitudinal stability. If it is sufficiently positive, then minor modeling discrepancies and computational errors will not affect the static longitudinal stability of wings level equilibrium flight.

*Remark #1:* The term $K_n$ is often referred to as the *stick-fixed* static margin, indicating that it is the static margin with the elevator locked in place. If the elevator is able to float freely under the influence of aerodynamic forces, it will tend to align itself with the flow and will generate no local lift force. The horizontal tail will therefore be less effective at providing stability. In these cases, one may also define a *stick-free static margin*, whose value will be somewhat smaller than the stick-fixed static margin. See the supplemental notes on hinge moments and stick forces.

*Remark #2:* Occasionally, pitch moment data will be given versus lift coefficient rather than angle of attack. In some sense, this makes the pitch stability question simpler because the zero-lift pitch moment (which must be positive) is more obvious from the data. If $C_m$ is given in terms of $C_L$, then we have $C_m(C_L(\alpha))$ and, by the chain rule,

$$\frac{d}{d\alpha} C_m(C_L(\alpha)) = \frac{dC_m}{dC_L}\frac{dC_L}{d\alpha}.$$

Thus,

$$\frac{dC_m}{dC_L} = \frac{C_{m_\alpha}}{C_{L_\alpha}}$$

and we have

$$h_n = h - \frac{dC_m}{dC_L}.$$

This equation allows one to estimate the stick-fixed neutral point from $C_m$ versus $C_L$ data for the aircraft. (Note, however, the important discussion in Section 2.3 of [1]!)

## References

[1] B. Etkin and L. D. Reid. *Dynamics of Flight: Stability and Control.* John Wiley and Sons, New York, NY, third edition, 1996.