# Lecture 3: Component Effects on Static Pitch Moment

**Pitch moment contribution from the wing.** Suppose that the aerodynamic characteristics of a

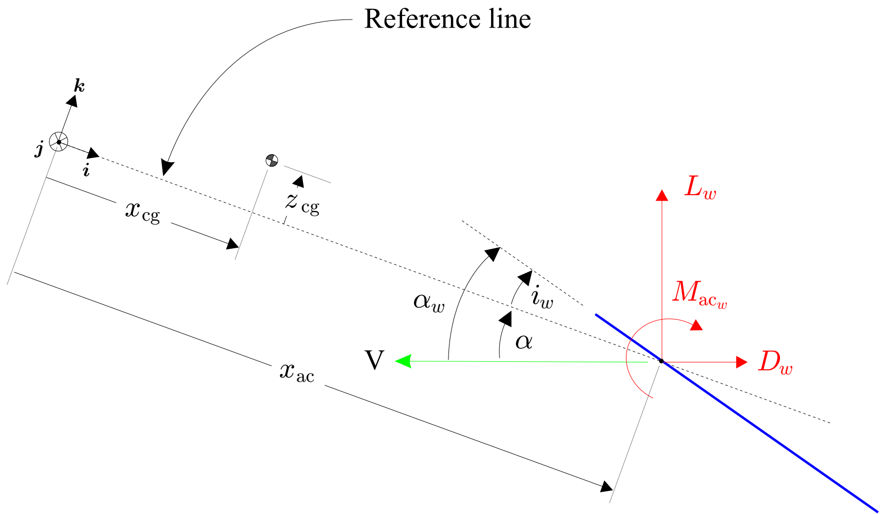


Figure 1: Longitudinal force and moment due to the wing.

particular airfoil on a particular aircraft are given with respect to the mean aerodynamic center. As in Figure 1, we replace the wing with a line denoting its mean aerodynamic chord of length $\bar{c}$. This line is drawn at angle of incidence $i_w$ with respect to a "fuselage reference line" which, we will assume, passes through the mean aerodynamic center. In longitudinal (wings-level) flight, the airflow about the wing generates two components of force and one component of moment:

$$\begin{aligned} \boldsymbol{L}_w &= L_w\left(-\sin\alpha\boldsymbol{i}+\cos\alpha\boldsymbol{k}\right) \\ \boldsymbol{D}_w &= D_w\left(\cos\alpha\boldsymbol{i}+\sin\alpha\boldsymbol{k}\right) \\ \boldsymbol{M}_{{\rm ac}_w} &= M_{{\rm ac}_w}\boldsymbol{j}. \end{aligned}$$

Keep in mind that $M_{{\rm ac}_w}$ remains constant as $\alpha$ varies. Choose the reference point (the origin of a body-fixed reference frame) to be some point on the reference line, as shown in Figure 1. The vector from the aircraft center of gravity to the aerodynamic center of the wing is

$$\begin{aligned} \boldsymbol{x}_{\rm ac/cg} &= \boldsymbol{x}_{\rm ac}-\boldsymbol{x}_{\rm cg} \\ &= (x_{\rm ac}\boldsymbol{i})-(x_{\rm cg}\boldsymbol{i}+z_{\rm cg}\boldsymbol{k}) \\ &= (x_{\rm ac}-x_{\rm cg})\boldsymbol{i}-z_{\rm cg}\boldsymbol{k} \end{aligned}$$

Summing the moments due to the wing about the aircraft center of gravity, we obtain

$$\begin{aligned} \boldsymbol{M}_{{\rm cg}_w} &= \boldsymbol{M}_{{\rm ac}_w}+\boldsymbol{x}_{\rm ac/cg}\times\boldsymbol{L}_w+\boldsymbol{x}_{\rm ac/cg}\times\boldsymbol{D}_w \\ &= \left[M_{{\rm ac}_w}+L_w\left(-(x_{\rm ac}-x_{\rm cg})\cos\alpha+z_{\rm cg}\sin\alpha\right)+D_w\left(-(x_{\rm ac}-x_{\rm cg})\sin\alpha-z_{\rm cg}\cos\alpha\right)\right]\boldsymbol{j}. \end{aligned}$$

Normalizing by $\frac{1}{2}\rho V^2 S\bar{c}$, we obtain

$$C_{m_{{\rm cg}_w}}=C_{m_{{\rm ac}_w}}+C_{L_w}\left[(h-h_{n_w})\cos\alpha+\frac{z_{\rm cg}}{\bar{c}}\sin\alpha\right]+C_{D_w}\left[(h-h_{n_w})\sin\alpha-\frac{z_{\rm cg}}{\bar{c}}\cos\alpha\right],$$

where $h=x_{\rm cg}/\bar{c}$ and $h_{n_w}=x_{\rm ac}/\bar{c}$.[1]

[1] The "$n$" in the term $h_{n_w}$ stands for *neutral point*, another term for aerodynamic center. The term arises from the pivotal role that the aerodynamic center plays in determining static longitudinal stability.

In normal operation, $\alpha \ll 1$ radian. Thus, in normal operation,

$$C_{m_{\mathrm{cg}_w}} \approx C_{m_{\mathrm{ac}_w}} + C_{L_w}\left[(h - h_{n_w}) + \frac{z_{\mathrm{cg}}}{\bar{c}}\alpha + \frac{C_{D_w}}{C_{L_w}}\left((h - h_{n_w})\,\alpha - \frac{z_{\mathrm{cg}}}{\bar{c}}\right)\right].$$

For a well-designed aircraft operating below stall, $\frac{C_{D_w}}{C_{L_w}} \ll 1$ (ignoring, once again, the pathological case where $C_{L_w} = 0$). Thus, the contribution of drag on the wing to pitch moment is often negligible. Typically, $z_{\mathrm{cg}}/\bar{c} \ll 1$, as well. Thus, for most aircraft in normal operation, we may write

$$\begin{aligned} C_{m_{\mathrm{cg}_w}} &\approx C_{m_{\mathrm{ac}_w}} + C_{L_w}(h - h_{n_w}) \\ &= C_{m_{\mathrm{ac}_w}} + \left(C_{L_{0_w}} + C_{L_{\alpha_w}}\alpha_w\right)(h - h_{n_w}) \end{aligned}$$

(You might recognize the first line as the moment transfer formula derived in Lecture 2.) Noting that $\alpha_w = \alpha + i_w$, we have

$$C_{m_{\mathrm{cg}_w}} = C_{m_{0_w}} + C_{m_{\alpha_w}}\alpha$$

where

$$C_{m_{0_w}} = C_{m_{\mathrm{ac}_w}} + \left(C_{L_{0_w}} + C_{L_{\alpha_w}} i_w\right)(h - h_{n_w})$$

and

$$C_{m_{\alpha_w}} = C_{L_{\alpha_w}}(h - h_{n_w}).$$

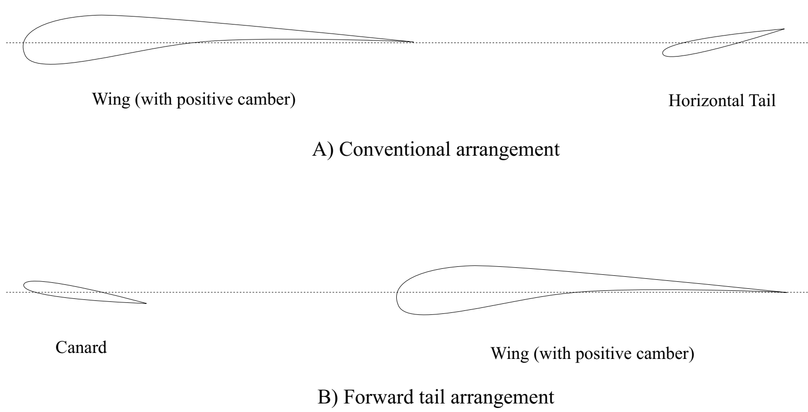


Figure 2: Conventional and forward tail arrangement.

Note that $C_{m_{0_w}} \neq C_{m_{\mathrm{ac}_w}}$ unless $\left(C_{L_{0_w}} + C_{L_{\alpha_w}} i_w\right) = 0$, that is, unless the fuselage reference line coincides with the zero-lift line of the wing. While this is not generally the case, let us assume that it is true for the sake of the following discussion. (Note: When they discuss the wing's contribution to pitch moment, Etkin and Reid *do* assume that the fuselage reference line coincides with the zero-lift line of the wing.) As the CG moves forward, $C_{m_{\alpha_w}}$ becomes more negative but $C_{m_{0_w}} = C_{m_{\mathrm{ac}_w}}$ remains constant. Treating the wing as the aircraft, recall that the two conditions for static pitch stability are:

$$C_{m_{\alpha_w}} < 0 \qquad \text{and} \qquad C_{m_{0_w}} > 0.$$

The former condition will be satisfied provided the CG is forward of the wing's aerodynamic center. The latter condition can be satisfied by an airfoil with negative camber, however wings with negative camber are less efficient at producing lift and can introduce problems with dynamic stability. In fact, most aircraft use airfoils with positive camber which typically results in $C_{m_{0_w}} < 0$. Thus, most conventionally used airfoils can not provide balanced flight without the addition of a horizontal tail or canard. See Figure 2, which is adapted from [2].

**Pitch moment contribution from the fuselage.** According to Etkin and Reid, the typical effect of the fuselage and nacelles on longitudinal static stability is

- a shift forward of the mean aerodynamic center,
- an increase in the lift-curve slope, and
- a negative increment in $C_{m_{\text{ac}}}$.

Although the flow around the wings, fuselage, and nacelles is generally quite complicated, there are techniques for estimating the contributions of these components to the pitch moment acting on the aircraft. See [3] for some simple techniques. See [1] for more sophisticated techniques.

Typically, the combined effects of the body and wing are incorporated into a single "wing-body" contribution to the pitch moment. Let $\alpha_{wb}$ denote the angle of attack measured from the zero-lift line of the combined wing and body. Then we may write

$$\begin{aligned} C_{m_{\text{cg}_{wb}}} &= C_{m_{\text{ac}_{wb}}} + C_{m_{\alpha_{wb}}} \alpha_{wb} \\ &= C_{m_{\text{ac}_{wb}}} + (h - h_{n_{wb}}) C_{L_{\alpha_{wb}}} \alpha_{wb}, \end{aligned}$$

where $h_{n_{wb}}$ is the neutral point of the combined wing and body, that is, the point about which the aerodynamic moment acting on the combined wing and body does not vary with angle of attack. The value of this constant moment is given by the coefficient $C_{m_{\text{ac}_{wb}}}$.

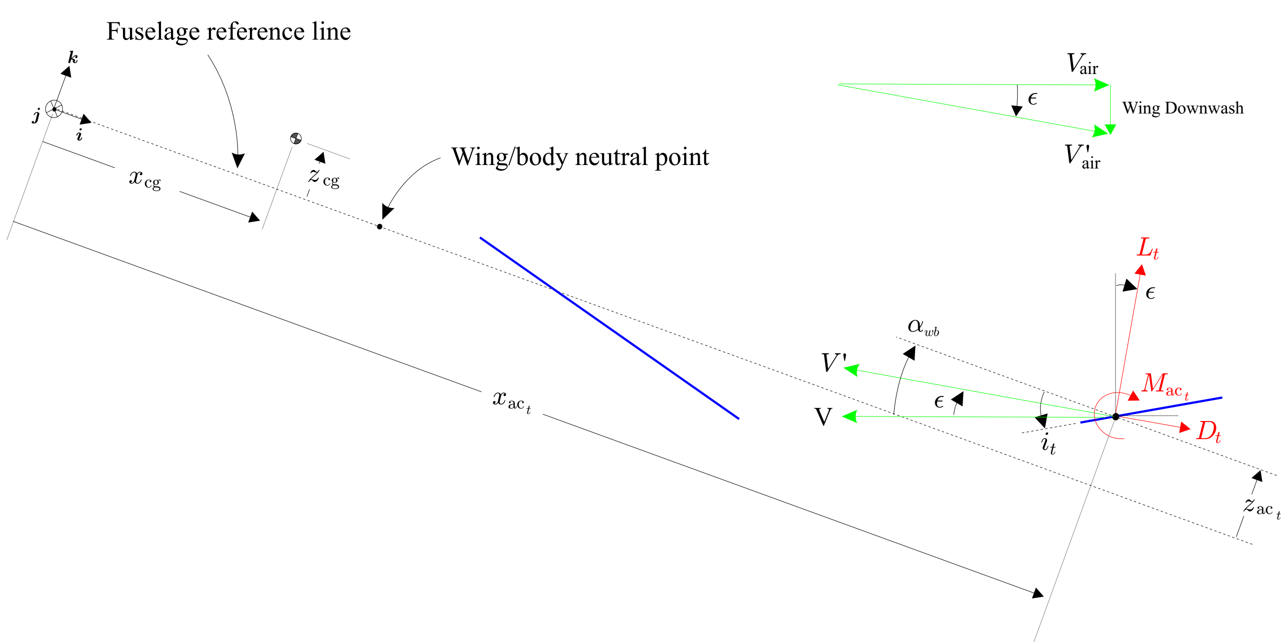


Figure 3: Longitudinal force and moment due to the tail.

**Pitch moment contribution from the horizontal tail.** The conventional way to address the problem of making $C_{m_{0_w}} > 0$ for an aircraft with a positively cambered wing is to include a horizontal tail, either aft or forward of the wing at an appropriate angle of incidence. Here, we consider the more conventional case of an aft tail. The case of a canard can be treated similarly.

The effect of an aft tail on the aircraft's longitudinal static stability can be treated very similarly to the effect of the wing, with at least two important differences. First, and more importantly, the *downwash* behind the wing *lowers* the effective angle of attack seen by the horizontal tail. Second, the local dynamic pressure at the wing may be reduced (for example, in the wake of the wing) or increased (for example, in the flow field behind a propeller or a jet engine). We will consider the former effect explicitly. The latter effect, which we will not discuss in depth, may be incorporated into the lift coefficient for the horizontal tail.

Referring to Figure 3, the angle of attack at the tail is

$$\alpha_t = \alpha_{wb} - \epsilon(\alpha_w) - i_t,$$

where $\epsilon(\alpha_w)$ is the downwash angle, a function of $\alpha_w$, and where $i_t$ is the incidence angle of the tail.[2] The lift and drag force generated by the tail are defined relative to the *local* velocity $\boldsymbol{V}'$ rather than the aircraft velocity $\boldsymbol{V}$. To combine these forces with the total aircraft lift and drag, they must be referred to the aircraft velocity $\boldsymbol{V}$. The angle between $\boldsymbol{V}'$ and $\boldsymbol{V}$ is due to the downwash aft of the main wing. This downwash is often described as being "induced" by the system of bound and trailing vortices which describe the circulation about a wing. Note that, forward of the wing, this system of vortices induces an upwash. (See Figure 4.) Thus, while the effective angle of attack at an aft tail is smaller than the wing angle of attack, the opposite is true for a canard.

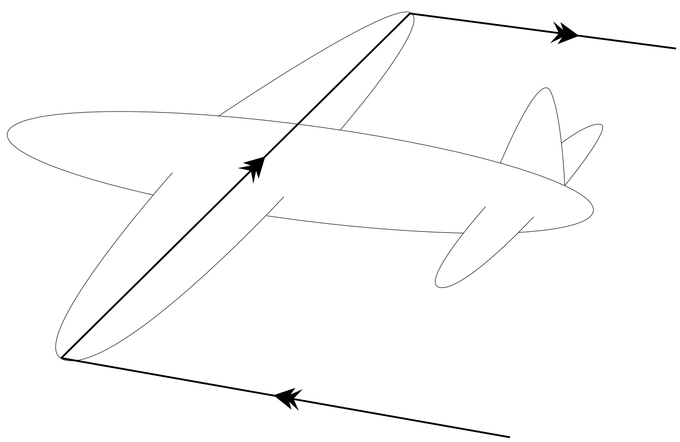

Figure 4: Horseshoe vortex representing circulation due to a wing.

The combined lift due to the wing-body and tail is

$$L = L_{wb} + L_t \cos\epsilon - D_t \sin\epsilon$$

As $\frac{D_t}{L_t}$ and $\epsilon$ are typically small, we may approximate the total lift as

$$L = L_{wb} + L_t = \left(C_{L_w} + \frac{S_t}{S} C_{L_t}\right)\left(\frac{1}{2}\rho V^2\right) S$$

where $S_t$ is the tail area. Now, the dynamic pressure at the tail is actually somewhat different than $\frac{1}{2}\rho V^2$ because the local velocity at the tail is different. The lift coefficient for the tail $C_{L_t}$ is therefore determined not only by its own geometry, but also by the effect of downwash. Some textbooks include a *tail efficiency* factor to account for the change in dynamic pressure at the tail. Etkin and Reid incorporate the effect directly into the lift coefficient $C_{L_t}$.

[2]Per Etkin and Reid's notation, the tail incidence angle is defined to be positive with the leading edge down.

The pitch moment about the aircraft CG due to the tail may be computed in a similar way to the wing's contribution. First, define

$$\begin{aligned}
\boldsymbol{L}_t &= L_t\left(-\sin(\alpha_{wb}-\epsilon)\boldsymbol{i}+\cos(\alpha_{wb}-\epsilon)\boldsymbol{k}\right)\\
\boldsymbol{D}_t &= D_t\left(\cos(\alpha_{wb}-\epsilon)\boldsymbol{i}+\sin(\alpha_{wb}-\epsilon)\boldsymbol{k}\right)\\
\boldsymbol{M}_{\mathrm{ac}_t} &= M_{\mathrm{ac}_t}\boldsymbol{j}.
\end{aligned}$$

The vector from the aircraft center of gravity to the aerodynamic center of the tail is

$$\begin{aligned}
\boldsymbol{x}_{\mathrm{ac}_t/\mathrm{cg}} &= \boldsymbol{x}_{\mathrm{ac}_t}-\boldsymbol{x}_{\mathrm{cg}}\\
&= (x_{\mathrm{ac}_t}\boldsymbol{i}+z_{\mathrm{ac}_t}\boldsymbol{k})-(x_{\mathrm{cg}}\boldsymbol{i}+z_{\mathrm{cg}}\boldsymbol{k})\\
&= (x_{\mathrm{ac}_t}-x_{\mathrm{cg}})\boldsymbol{i}+(z_{\mathrm{ac}_t}-z_{\mathrm{cg}})\boldsymbol{k}\\
&= l_t\boldsymbol{i}+z_t\boldsymbol{k}
\end{aligned}$$

where

$$l_t = x_{\mathrm{ac}_t}-x_{\mathrm{cg}} \qquad \text{and} \qquad z_t = z_{\mathrm{ac}_t}-z_{\mathrm{cg}}$$

denote the *signed* horizontal and vertical distance from the CG to the mean aerodynamic center of the tail.[3] Summing the moments due to the tail about the aircraft center of gravity, we obtain

$$\begin{aligned}
\sum \boldsymbol{M}_{\mathrm{cg}_t} &= M_{\mathrm{cg}_t}\boldsymbol{j}\\
&= \boldsymbol{M}_{\mathrm{ac}_t}+\boldsymbol{x}_{\mathrm{ac}_t/\mathrm{cg}}\times\boldsymbol{L}_t+\boldsymbol{x}_{\mathrm{ac}_t/\mathrm{cg}}\times\boldsymbol{D}_t\\
&= \Big[M_{\mathrm{ac}_t}+L_t\left(-l_t\cos(\alpha_{wb}-\epsilon)-z_t\sin(\alpha_{wb}-\epsilon)\right)\\
&\qquad +D_t\left(-l_t\sin(\alpha_{wb}-\epsilon)+z_t\cos(\alpha_{wb}-\epsilon)\right)\Big]\boldsymbol{j}.
\end{aligned}$$

Now, since

$$L_t = C_{L_t}\left(\frac{1}{2}\rho V^2\right)S_t \qquad \text{and} \qquad D_t = C_{D_t}\left(\frac{1}{2}\rho V^2\right)S_t,$$

we obtain

$$\begin{aligned}
M_{\mathrm{cg}_t} &= \left(\frac{1}{2}\rho V^2\right)S_t\Big[C_{m_{\mathrm{ac}_t}}\bar{c}_t+C_{L_t}\left(-l_t\cos(\alpha_{wb}-\epsilon)-z_t\sin(\alpha_{wb}-\epsilon)\right)\\
&\qquad +C_{D_t}\left(-l_t\sin(\alpha_{wb}-\epsilon)+z_t\cos(\alpha_{wb}-\epsilon)\right)\Big].
\end{aligned}$$

Normalizing the tail moment equation, we obtain

$$\begin{aligned}
C_{m_t} &= \frac{S_t}{S}\Bigg[C_{m_{\mathrm{ac}_t}}\frac{\bar{c}_t}{\bar{c}}+C_{L_t}\left(-\frac{l_t}{\bar{c}}\cos(\alpha_{wb}-\epsilon)-\frac{z_t}{\bar{c}}\sin(\alpha_{wb}-\epsilon)\right)\\
&\qquad +C_{D_t}\left(-\frac{l_t}{\bar{c}}\sin(\alpha_{wb}-\epsilon)+\frac{z_t}{\bar{c}}\cos(\alpha_{wb}-\epsilon)\right)\Bigg]\\
&= \frac{S_t\bar{c}_t}{S\bar{c}}C_{m_{\mathrm{ac}_t}}+\frac{S_t l_t}{S\bar{c}}C_{L_t}\Bigg[\left(-\cos(\alpha_{wb}-\epsilon)-\frac{z_t}{l_t}\sin(\alpha_{wb}-\epsilon)\right)\\
&\qquad +\frac{C_{D_t}}{C_{L_t}}\left(-\sin(\alpha_{wb}-\epsilon)+\frac{z_t}{l_t}\cos(\alpha_{wb}-\epsilon)\right)\Bigg]
\end{aligned}$$

[3] The definition $l_t$ is consistent with that given by Etkin and Reid. The definition of $z_t$ is *not*; there is a sign discrepancy due to the authors' inconsistent use of the coordinate $z$.

Now, in normal operation, $(\alpha_{wb} - \epsilon)$ and $C_{D_t}/C_{L_t}$ are small. Also, $C_{m_{\text{ac}_t}} \frac{\bar{c}_t}{\bar{c}}$ is typically small in comparison to the other terms; if the tail airfoil is symmetric, for example, then $C_{m_{\text{ac}_t}} = 0$. Terms involving $z_t$ are typically ignored, as well. It follows that, in estimating the pitch moment contribution of the tail, it is often reasonable to consider only the contribution of the tail's lift acting through the moment arm $l_t$. We therefore obtain

$$C_{m_t} \approx -\left(\frac{S_t l_t}{S\bar{c}}\right) C_{L_t}$$

The term in parentheses plays such an important role in design that we give it a special name and symbol. The *horizontal tail volume ratio*

$$V\!\!\!-_{\text{H}} = \frac{S_t l_t}{S\bar{c}}$$

is an indicator of the horizontal tail's effectiveness at generating a pitch moment. For example, a horizontal tail with a large planform area $S_t$ acting through a large moment arm $l_t$ will be very effective at generating a restoring pitch moment in response to pitch disturbances. We therefore find that the contribution of the tail to the pitch moment about the CG is

$$\boxed{C_{m_t} = -V\!\!\!-_{\text{H}} C_{L_{\alpha_t}} \alpha_t}$$

Etkin and Reid note that the distance $l_t$ from the aircraft CG to the tail aerodynamic center varies with the aircraft's mass distribution. For example, the CG location may change with different passenger or cargo loading arrangements or as fuel is spent. Because $l_t$ varies, the horizontal tail volume ratio also varies. We will assume, however, that this effect is small enough to be neglected.

## References


[1] *USAF Stability and Control Datcom.* Flight Control Division, Air Force Flight Dynamics Laboratory, Wright Patterson Air Force Base, Fairborn, OH.

[2] B. Etkin and L. D. Reid. *Dynamics of Flight: Stability and Control.* John Wiley and Sons, New York, NY, third edition, 1996.

[3] R. C. Nelson. *Flight Stability and Automatic Control.* WCB McGraw-Hill, New York, NY, second edition, 1998.