Lecture 24: Root Locus Diagrams

Absolute and Relative Stability. A LTI control system is called *absolutely stable* if the controlled transfer function $H_d(s)$ from the reference signal $y_d(s)$ to the output signal $y(s)$ has all of its poles in the open left half plane. One technique for determining absolute stability of a control system is the Routh-Hurwitz stability analysis technique. This very useful technique is presented in Section 5-7 of [1].

Absolute stability is an essential quality for a control system, but it says nothing about the performance characteristics of the system, i.e., the transient response. Two "absolutely stable" systems can respond to a step input in very different ways; one might exhibit a very slow, overdamped response while the other exhibits a very fast, underdamped response.

To compare the performance of two absolutely stable systems, it is useful to consider the notion of "relative stability" or "degree of stability." Degree of stability can be rather narrowly defined as the horizontal distance between the imaginary axis and the nearest pole. This distance will typically determine the speed of response of the system, however it tells you nothing more about the nature of that response (e.g., if it the system is overdamped, critically damped, or underdamped). More generally, one may examine the specific locations of the closed-loop poles. Knowing these pole locations gives a good sense of the nature of the system's transient response.

The Root Locus Method. The root locus method, also known as "Evans' rules" in honor of W. R. Evans, is a technique for determining how the poles of a feedback control system move in the complex plane as a parameter is varied. Typically, the parameter is a control gain, although any parameter of interest can be used. (For this reason, the root locus method is useful in dynamical system theory, where one is often interested in sudden changes in a system's qualitative behavior, called "bifurcations," as a parameter varies.)

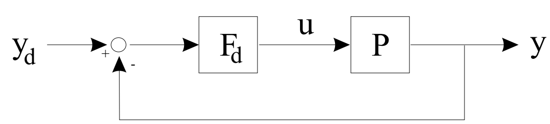

Figure 1: One degree of freedom closed-loop control structure.

Consider the simple feedback control system shown in Figure 1. The closed-loop transfer function is

$$H_d(s) = \frac{y(s)}{y_d(s)} = \frac{P(s)F_d(s)}{1 + P(s)F_d(s)}.$$

Closed-loop poles are values of s for which

$$1 + P(s)F_d(s) = 0.$$

Since $P(s)F_d(s)$ is a function of a complex variable, the equation $P(s)F_d(s) = -1$ can be expressed in terms of the magnitude and phase of $P(s)F_d(s)$:

$$|P(s)F_d(s)| = 1 \quad \text{and} \quad \angle P(s)F_d(s) = (2k+1)\pi \quad k = 0, \pm 1, \pm 2, \ldots$$

In words, the magnitude of the "loop gain" is always one and the phase is an odd power of π.

Suppose that $P(s)F_d(s)$ can be written in the form

$$P(s)F_d(s) = K\frac{b(s)}{a(s)}.$$

This would be the case, for example, if $P(s) = b(s)/a(s)$ and $F_d = K$, as for a simple proportional controller. The control structure might be more complicated than this, however we assume that a the multiplicative factor K appears and that this parameter may vary.

The "root locus" is the "locus" of possible roots of the closed-loop transfer function as the multiplicative parameter K is varied. In fact, the entire root locus can be determined from the angle condition alone. The magnitude condition is then used to determine which value of K corresponds to which set of closed-loop poles along the locus of all possible closed-loop poles.

Rather than learn Evans' rules to begin with, it is more illustrative to consider a series of increasingly complicated examples.

Example 1. To begin, we consider the very simple example

$$P(s)F_d(s) = K\frac{1}{s(s+2)}.$$

We will compute the closed-loop poles as explicit functions of K. In general, this is a tedious, and uninformative exercise, but for this simple system it serves to illustrate how closed-loop poles vary as the gain K is varied. The closed-loop transfer function is

$$H_d(s) = \frac{\frac{K}{s(s+2)}}{1 + \frac{K}{s(s+2)}}.$$

The closed-loop poles are obtained from

$$\begin{aligned} 0 &= 1 + \frac{K}{s(s+2)} \\ &= s^2 + 2s + K. \end{aligned}$$

They are

$$\begin{aligned} s &= \frac{1}{2}(-2 \pm \sqrt{4-4K}) \\ &= -1 \pm \sqrt{1-K}. \end{aligned}$$

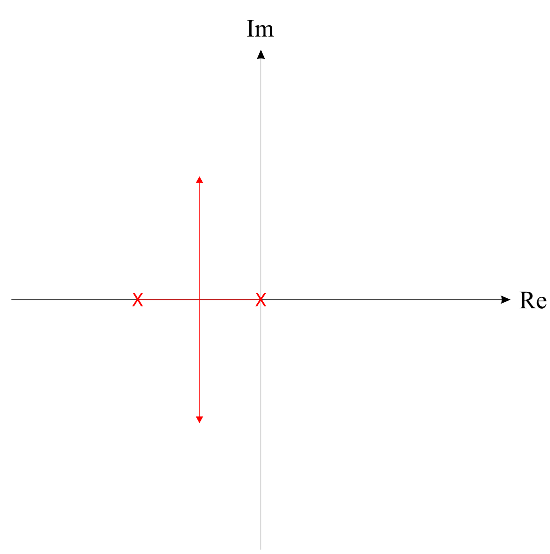

Figure 2: Root locus example #1.

When $0 < K < 1$, there are two distinct poles which are located on the real axis between 0 and -2. When $K = 1$, the poles coalesce at $s = -1$. As K continues to increase, the poles split apart and move in opposite directions parallel to the imaginary axis.

To see that the locus of closed-loop poles shown in Figure 2 can be obtained from the angle condition

$$\angle \frac{1}{s(s+2)} = (2k+1)\pi \quad k = 0, \pm 1, \pm 2, \ldots,$$

we first recall some facts about complex numbers. First, a complex number can be represented in polar form, for example $z = re^{i\theta}$ where r is the radial distance from the origin to the point z and θ is the angle to z measured counter-clockwise from the positive real axis. Consider the complex function

$$C(s) = \frac{(s - z_1)\cdots(s - z_m)}{(s - p_1)\cdots(s - p_n)}.$$

Each term in the numerator can be considered a vector from the zero z_i to the point s. Similarly, each term in the denominator can be considered a vector from the pole p_i to the point s. Each of these vectors has a magnitude and an angle, so we may equivalently write

$$\begin{aligned} C(s) &= \frac{\left(r_{z_1} e^{i\theta_{z_1}}\right)\cdots\left(r_{z_m} e^{i\theta_{z_m}}\right)}{\left(r_{p_1} e^{i\theta_{p_1}}\right)\cdots\left(r_{p_n} e^{i\theta_{p_n}}\right)} \\ &= \left(\frac{r_{z_1}\cdots r_{z_m}}{r_{p_1}\cdots r_{p_n}}\right) e^{i\left(\theta_{z_1}+\cdots+\theta_{z_m}-\theta_{p_1}-\cdots-\theta_{p_n}\right)} \end{aligned}$$

where r_{z_i} (or r_{p_i}) is the magnitude of the vector from z_i (or p_i) to s and θ_{z_i} (or θ_{p_i}) is the angle of the vector from z_i (or p_i) to s.

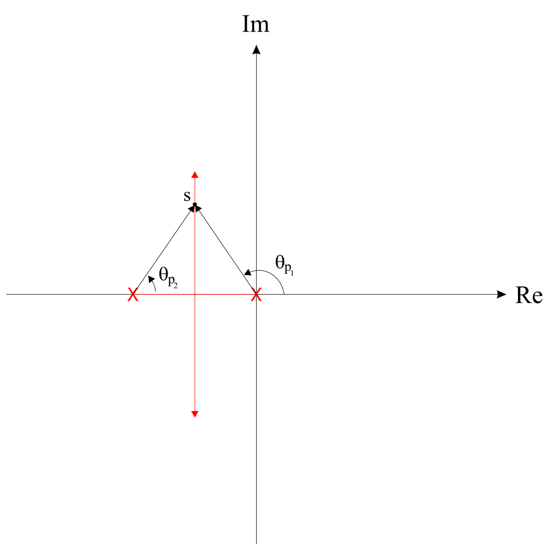

Figure 3: Angle condition for root locus example #1.

Applying these observations to the current example, we find that

$$\angle \frac{1}{s(s+2)} = -\angle s - \angle(s+2). \tag{1}$$

Now, for any point on the real axis to the right of $p_1 = 0$, equation (1) gives zero, which is *not* an odd number times π. Similarly, for any point on the real axis to the left of $p_2 = -2$, equation (1) gives -2π, which is also not an odd number times π. Thus, the real axis to the left of p_2 and to the right of p_1 is not part of the root locus. However, for points between p_2 and p_1, equation (1) gives

$$\angle \frac{1}{s(s+2)} = -\angle s - \angle(s+2) = -\pi - 0,$$

which *is* an odd number times π. Thus, points on the real axis between p_2 and p_1 are part of the root locus.

Considering next the points on the vertical line $s = -1$, we choose a point and determine $\angle \frac{1}{s(s+2)}$. The vectors from p_1 and p_2 to any such point form an isosceles triangle. The sum of the two angles is π for points above the real axis and 3π for points below the real axis, giving $\angle \frac{1}{s(s+2)} = -\pi$ or -3π, respectively. Thus, the line $s = -1$ is part of the root locus.

To find the value of K which corresponds to a particular pair of closed-loop poles, we use the magnitude condition. For example, suppose we would like to choose K so that the closed-loop system has a damping ratio $\zeta = \frac{\sqrt{2}}{2}$. Any pole lying on the radius $\theta = \frac{3\pi}{4}$ in the complex plane has damping ratio $\zeta = \frac{\sqrt{2}}{2}$. Thus, we would like to choose K to give closed-loop poles at

$$s = -1 \pm i \tan \frac{\pi}{4} = -1 \pm i.$$

Choosing a particular pole, say $s = -1 + i$, we substitute into the magnitude condition to obtain

$$\left| \frac{K}{(-1+i)((-1+i)+2)} \right| = 1$$

or

$$K = |(-1+i)(1+i)| = |-2| = 2.$$

Thus, choosing the gain $K = 2$ gives the closed-loop poles $s = -1 \pm i$. $\square$

We have assumed that $P(s)F_d(s)$ can be written in the form

$$P(s)F_d(s) = K\frac{b(s)}{a(s)}.$$

where $b(s)$ has degree m, $a(s)$ has degree $n \geq m$ and where $K > 0$ is a parameter (e.g., a control gain) which may vary.

An important observation is that, as $K \to 0$, the closed-loop poles approach the poles of the loop gain. To see this, write the closed-loop characteristic equation as

$$a(s) + Kb(s) = 0.$$

Clearly, as $K \to 0$ the roots of the polynomial on the left approach the roots of $a(s)$.

One may also observe that, as $K \to \infty$, the closed-loop poles must either diverge to ∞ or approach a zero of the loop gain. To see this, recognize that as $K \to \infty$, $\frac{b(s)}{a(s)}$ must become very small so that the product is -1. There are two ways that $\frac{b(s)}{a(s)}$ can become very small. First, $b(s)$ can go to zero (which happens when s approaches a zero of the loop gain). Second, $a(s)$ can go to infinity (which can only happen when $|s|$ goes to infinity.) In general, m branches of the root locus approach the zeros of the loop gain while the remaining $n - m$ branches go to infinity.

Example 2. Consider the following example from [1]:

$$P(s)F_d(s) = \frac{K}{s(s+1)(s+2)}.$$

This system has poles at $p_1 = 0$, $p_2 = -1$, and $p_3 = -2$. Recalling that

$$\angle P(s)F_d(s) = \sum_{i=1}^{m} \angle(s - z_i) - \sum_{j=1}^{n} \angle(s - p_j),$$

we first consider which, if any, points on the real axis are part of the root locus. For any point to the right of $s = 0$, $\angle P(s)F_d(s) = 0$, so the positive real axis is *not* part of the root locus. For any point $-1 < s < 0$, $\angle P(s)F_d(s) = -\pi$, so these points *are* part of the root locus. For any point $-2 < s < -1$, $\angle P(s)F_d(s) = -2\pi$, so these points are *not* part of the root locus. Finally, for any point $s < -2$, $\angle P(s)F_d(s) = -3\pi$, so these points *are* part of the root locus.

Next, we consider what happens to the root locus as s grows large. In the limit that s grows large, we have

$$\lim_{|s|\to\infty} P(s)F_d(s) = \lim_{|s|\to\infty} \frac{K}{s(s+1)(s+2)} = \lim_{|s|\to\infty} \frac{K}{s^3}.$$

Now, no matter how large $|s|$ is, the angle condition must be satisfied, so we must have

$$\begin{aligned}
\lim_{|s|\to\infty} \angle\, P(s)F_d(s) &= \lim_{r\to\infty} \angle\, P(re^{i\theta})F_d(re^{i\theta}) \\
&\approx \lim_{r\to\infty} \frac{K}{(re^{i\theta})^3} \\
&= \angle\, e^{-i3\theta} \\
&= (2n+1)\pi \qquad n = 0, \pm1, \pm2, \ldots
\end{aligned}$$

or

$$\theta = -\frac{2n+1}{3}\pi.$$

Trying $n = 0$ gives $\theta = -\frac{\pi}{3}$. Trying $n = 1$ gives $\theta = -\pi$. Trying $n = 2$ gives $\theta = -\frac{5\pi}{3}$. Other choices of n give repeated angles. In the limit that $|s| \to \infty$, the three closed-loop poles follow asymptotes that extend radially in the directions $\pm\frac{\pi}{3}$ and π.

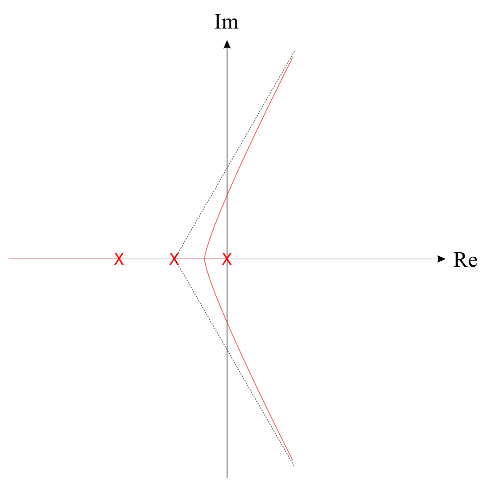

Figure 4: Root locus example #2.

Two of the three asymptotes extend into the right half complex plane, while the third follows the negative real axis. Intuitively, the closed-loop pole which starts (for small K) at $s = -2$ will follow the negative real axis asymptote as K increases. Therefore, the two closed-loop poles which rest on the real axis between $s = -1$ and $s = 0$ must coalesce and split off to follow the asymptotes at $\pm\frac{\pi}{3}$.[1]

[1] They must first coalesce because poles must be either real numbers or complex conjugate pairs and because the closed-loop pole locations vary continuously with K.

Note: You can find the value of the gain K at which the root locus passes into the right half plane by performing a Routh-Hurwitz stability analysis and finding conditions on K for stability.

References

[1] K. Ogata. *Modern Control Engineering, Fourth Ed.* Prentice Hall, Upper Saddle River, NJ, 2002.