Lecture 23: Routh-Hurwitz Stability Analysis

The following discussion follows that of [2]. For more detailed information about Routh-Hurwitz stability analysis, see [1].

Suppose we have a transfer function

$$Q(s) = \frac{B(s)}{A(s)}.$$

The transfer function $Q(s)$ might represent the dynamics of an uncontrolled plant, an open-loop controlled system, or a closed-loop (feedback controlled) system. In any case, $Q(s)$ describes the relationship between an input, say $U(s)$, and an output, say $X(s)$. For example, the system response to a unit step input can be obtained as

$$x(t) = \mathcal{L}^{-1}\left[X(s)\right] = \mathcal{L}^{-1}\left[Q(s)\frac{1}{s}\right].$$

If *any* root of $A(s)$ has positive real part, then the system (or, equivalently, the transfer function) is said to be unstable. As a result $x(t)$ may grow large with time. This is clearly unacceptable if one is attempting to drive x to some desired value x_d. Absolute stability (i.e., having all characteristic values in the left half complex plane) is a fundamental requirement for a control system.

The problem is that $A(s)$ may be very high order, so finding its roots explicitly (in order to check that they have negative real part) could be difficult. The quadratic formula works for second order polynomials. Formulae exist up to order *five*, but they are very messy and numerical methods are more often used for third and higher degree polynomials. *Routh's stability criterion* provides a simple algorithm for determining whether a system is unstable *without* solving for the roots explicitly.

Routh's Stability Algorithm.
Step 1. Write the polynomial in the form

$$A(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n.$$

One may assume that a_n is nonzero. Otherwise, there is a root at $s = 0$ and one may divide $A(s)$ by s to check the remaining roots. A necessary (but not sufficient) condition for $A(s)$ to be *Hurwitz*, i.e., for all of its roots to have strictly negative real part, is that all coefficients be positive. If any coefficient is zero or negative, then there is at least one root in the closed right-half plane.

Step 2. If all coefficients are positive, form the "Routh Array" as follows

$$\begin{array}{c|cccc}
s^n & a_0 & a_2 & a_4 & \cdots \\
s^{n-1} & a_1 & a_3 & a_5 & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \\
s^2 & d_1 & d_2 & & \\
s^1 & e_1 & & & \\
s^0 & f_1 & & &
\end{array}$$

The powers of s to the left serve as "place-keepers" to remind you how many rows of the array remain to be computed. (More in a moment.)

The third row of the array is computed according to the formula

$$b_i = \frac{a_1 a_{2i} - a_0 a_{2i+1}}{a_1}.$$

Start with $i = 1$ and iterate until $b_i = 0$ for all higher values of i.

The fourth row of the array is computed according to the formula

$$c_i = \frac{b_1 a_{2i+1} - a_1 b_{i+1}}{b_1}.$$

Start with $i = 1$ and iterate until $c_i = 0$ for all higher values of i.

Continue until f_1 has been computed.

Step 3. Apply *Routh's Stability Criterion* to determine the number of roots with positive real part:

Routh's Stability Criterion: The number of roots with positive real part is equal to the number of sign changes in the left-most column of the Routh array.

Example: Determine how many roots of the polynomial

$$3s^6 + s^5 + 2s^3 + s^2 + 5s + 1$$

have positive real part. We may rewrite this polynomial as

$$3s^6 + 1s^5 + 0s^4 + 2s^3 + 1s^2 + 5s + 1$$

The Routh array may be computed as

$$\begin{array}{lcccc}
s^6 & 3 & 0 & 1 & 1 \\
s^5 & 1 & 2 & 5 & \\
s^4 & \frac{1\cdot 0 - 3\cdot 2}{1} & \frac{1\cdot 1 - 3\cdot 5}{1} & \frac{1\cdot 1 - 3\cdot 0}{1} & \\
 & -6 & -14 & 1 & \\
s^3 & \frac{-6\cdot 2 - 1\cdot(-14)}{-6} & \frac{-6\cdot 5 - 1\cdot 1}{-6} & & \\
 & -\frac{1}{3} & \frac{31}{6} & & \\
 & -2 & 31 & & \\
s^2 & \frac{(-2)\cdot(-14) - (-6)\cdot 31}{-2} & \frac{(-2)\cdot 1 - (-6)\cdot 0}{-2} & & \\
 & -107 & 1 & & \\
s^1 & \frac{(-107)\cdot 31 - (-2)\cdot 1}{-107} & & & \\
 & \frac{3315}{107} & & & \\
s^0 & 1 & & &
\end{array}$$

Because there are *two* sign changes in the left-most column, there are two roots with positive real part.

Remark: If the sign of all coefficients in the polynomial $A(s)$ is not the same, then at least one root of $A(s)$ has positive real part. If determining stability is the only objective, one may stop here! (However, if one wants to know the precise number of unstable roots, one must proceed with the algorithm as described.)

Remark: Routh's stability criterion describes how many roots have *positive* real part. The remaining roots may have negative *or zero* real part. *A necessary and sufficient condition for all roots to have strictly negative real part is that all coefficients in the polynomial $A(s)$ are strictly positive (no "zero" coefficients!) and no sign changes occur in the left-most column of the Routh array.*

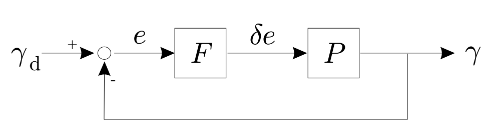

Figure 1: The closed-loop control system for Problem 1.

Example: Flight path regulator. Recall the STOL aircraft example we considered in the previous lecture. The transfer function from elevator to flight path angle is

$$P(s) = \frac{\gamma(s)}{\delta e(s)} \approx \frac{7s^3 + 0.1s^2 - 1000s - 8}{s^4 + 5s^3 + 9s^2 + 0.2s + 0.06}.$$

(For simplicity, we have only retained a single significant digit in the polynomial coefficients.) With proportional error feedback

$$F(s) = \frac{\delta e(s)}{e(s)} = k_p,$$

where $e = \gamma_{\mathrm{d}} - \gamma$, the closed-loop transfer function is

$$\begin{aligned} H(s) = \frac{\gamma(s)}{\gamma_{\mathrm{d}}(s)} &= \frac{P(s)F(s)}{1 + P(s)F(s)} \\ &= \frac{k_p\left(7s^3 + 0.1s^2 - 1000s - 8\right)}{s^4 + (5 + 7k_p)\,s^3 + (9 + 0.1k_p)\,s^2 + (0.2 - 1000k_p)\,s + (0.06 - 8k_p)} \end{aligned}$$

We previously obtained the following necessary conditions on the proportional gain k_p:

$$-\frac{4.7}{7} < k_p < \frac{0.2}{1000}.$$

To obtain necessary and sufficient conditions, we apply the Routh-Hurwitz stability analysis technique.

$$\begin{array}{cccc}
s^4 & 1 & 9 + 0.1k_p & 0.06 - 8k_p \\
s^3 & 5 + 7k_p & 0.2 - 1000k_p & \\
s^2 & \frac{(5+7k_p)(9+0.1k_p)-(1)(0.2-1000k_p)}{(5+7k_p)} & \frac{(5+7k_p)(0.06-8k_p)-(1)(0)}{(5+7k_p)} & \\
 & \frac{0.7k_p^2+1000k_p+50}{5+7k_p} & 0.06 - 8k_p & \\
s^1 & \frac{-7E3k_p^3-1E6k_p^2-5E4k_p+10}{0.7k_p^2+1000k_p+50} & & \\
s^0 & 0.06 - 8k_p & &
\end{array}$$

Stability requires not only the previous linear inequalities, but also that

$$0.7k_p^2 + 1000k_p + 50 > 0 \qquad \text{and} \qquad -7E3k_p^3 - 1E6k_p^2 - 5E4k_p + 10 > 0.$$

Thus, the conditions on k_p are considerably more complicated than simple linear inequalities. The end result is that only a very tiny range of proportional gains $k_p > 0$ will ensure closed-loop stability.

It is evident that controlling flight path angle using elevator alone is a challenging control design problem. Of course, there are more controls available than just the elevator and control design should logically take this into consideration. The traditional approach, however, has been to break the system into a set of critical "single-input, single-output (SISO)" channels and to consider them one at a time. At the end, the control designer verifies that the complete, closed-loop "multi-input, multi-output (MIMO)" system is well-behaved. An alternative to this "classical" approach is to use "modern control theory" to design a regulator for the complete MIMO system. To learn more about classical and modern control techniques, you may be interested in taking AOE 4004 the next time it is offered.

The Routh-Hurwitz approach is a method of assessing *absolute stability.* It provides necessary and sufficient conditions for all of the characteristic values to have strictly negative real part. It does not, however, indicate anything else about the closed-loop characteristic values. Recall that aircraft handling quality specifications give desired ranges for damping ratio and natural frequency for the characteristic of modes of motion. These specifications can be translated into specific desired pole locations for the feedback-controlled system. To study this problem of *relative stability*, one must use a tool such as the root locus method, which shows how closed-loop characteristic values move in the complex plane as a control gain is varied. Again, to learn more about classical and modern control techniques, consider taking AOE 4004 the next time it is offered.

Appendix: Special Cases in Routh-Hurwitz Analysis

A zero in the left-most column. If there is a zero in the left-most column, the procedure cannot be applied as stated because one must divide by zero. To avoid this problem, replace the zero with a *small, positive constant* ϵ and proceed as before.

- If the sign changes from the row *before* ϵ to the row *after* ϵ, then this indicates one root with positive real part.
- If the sign does not change from the row *before* ϵ to the row *after* ϵ, then there is a complex pair of roots.

Example: A root with positive real part exists if,

$$\begin{array}{ccc} s^{i+1} & 5 & \cdots \\ s^i & \epsilon & \cdots \\ s^{i-1} & -2 & \cdots \end{array} \qquad \text{or} \qquad \begin{array}{ccc} s^{i+1} & -1 & \cdots \\ s^i & \epsilon & \cdots \\ s^{i-1} & 1 & \cdots \end{array}$$

A pure imaginary pair of roots exists if,

$$\begin{array}{ccc} s^{i+1} & 5 & \cdots \\ s^i & \epsilon & \cdots \\ s^{i-1} & 2 & \cdots \end{array} \qquad \text{or} \qquad \begin{array}{ccc} s^{i+1} & -1 & \cdots \\ s^i & \epsilon & \cdots \\ s^{i-1} & -1 & \cdots \end{array}$$

Example: Determine how many roots of the polynomial

$$s^3 + s + 10$$

have positive real part. We may rewrite this polynomial as

$$1s^3 + 0s^2 + 1s + 10$$

The Routh array may be computed as

$$\begin{array}{ccc} s^3 & 1 & 1 \\ s^2 & 0 & 10 \\ & \epsilon & 10 \\ s^1 & \frac{\epsilon \cdot 1 - 1 \cdot 10}{\epsilon} & \\ & 1 - \frac{10}{\epsilon} & \\ s^0 & 1 & \end{array}$$

Because $0 < \epsilon \ll 1$, the third element in the left-most column is *negative.* Thus, there are *two* sign changes and therefore two roots with positive real part. One may verify that the roots are

$$s = -2,\ 1 \pm 2j.$$

An entire row of zeros. An entire row of zeros indicates a symmetry in the distribution of roots. To continue with the procedure, one replaces this row of the Routh array using coefficients from the derivative of an *auxiliary polynomial.*

Example: Determine how many roots of the polynomial

$$s^5 + 2s^4 + s + 2.$$

have positive real part. We may rewrite this polynomial as

$$1s^5 + 2s^4 + 0s^3 + 0s^2 + 1s + 2.$$

The Routh array terminates too early with a "zero row"

$$\begin{array}{llll} s^5 & 1 & 0 & 1 \\ s^4 & 2 & 0 & 2 \\ s^3 & 0 & \frac{2\cdot 1 - 1\cdot 2}{2} & \\ & 0 & 0 & \\ s^2 & & & \\ s^1 & & & \\ s^0 & & & \end{array}$$

The auxiliary polynomial $P(s)$ is formed from the row *preceding* the zero row:

$$\begin{aligned} P(s) &= 2s^4 + 0s^2 + 2 \\ &= 2(s^4 + 1). \end{aligned}$$

As an aside, note that four of the roots of the original polynomial may be obtained from the roots of $P(s)$. To continue with the Routh procedure, take the derivative of $P(s)$:

$$\frac{d}{ds}P(s) = 8s^3.$$

We may continue the Routh array by substituting the coefficients of $\frac{dP}{ds}$.

$$\begin{array}{llll} s^5 & 1 & 0 & 1 \\ s^4 & 2 & 0 & 2 \\ s^3 & 8 & 0 & \\ s^2 & 0 & \frac{8\cdot 2 - 2\cdot 0}{8} & \\ & \epsilon & 2 & \\ s^1 & \frac{\epsilon\cdot 0 - 8\cdot 2}{\epsilon} & & \\ & -\frac{16}{\epsilon} & & \\ s^0 & 2 & & \end{array}$$

There are *two* sign changes so there are two roots with positive real part. One may check that the roots are

$$s = -2, \ \pm 1 \pm j$$

References

[1] L. Cesari. *Asymptotic Behavior and Stability Problems in Ordinary Differential Equations.* Springer-Verlag, New York, 3rd edition, 1971.

[2] K. Ogata. *Modern Control Engineering.* Prentice Hall, Englewood Cliffs, NJ, 4th edition.