Lecture 22: A Brief Introduction to Linear Control

Figure 1, which is taken from [1], represents the general form of a control system. The fundamental component is the plant itself. The "Plant Dynamics" block in Figure 1 represents some system whose behavior is influenced by the application of a control input $\boldsymbol{u}$. While the complete behavior of the plant is given by its state history, there may be some subset of state variables which are of particular interest to the control designer; these define the output $\boldsymbol{y}$. While we will consider only linear, time-invariant systems, this control system structure is valid for a much more general class of systems. For example, the dynamics in any of the blocks might be described by nonlinear, time-varying ODE's or even PDE's. Of course, the problem of control design and analysis is much more difficult in those cases.

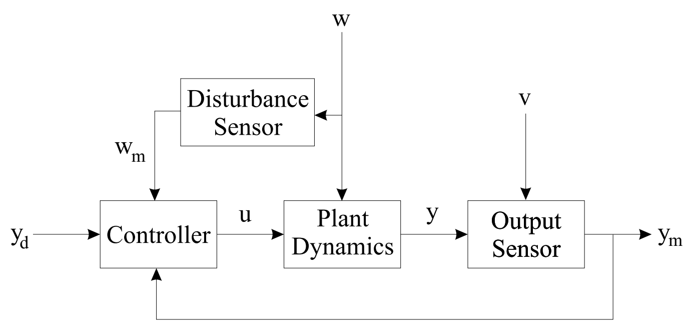

Figure 1: Functional diagram of a general control system.

Besides the input $\boldsymbol{u}$, the plant's behavior may also be influenced by a disturbance $\boldsymbol{w}$. In designing a controller, it is necessary to consider the nature of the disturbances which will influence the system. Sometimes, in order to meet performance objectives, it is even necessary to measure the disturbances explicitly and "pre-compensate" for them. In this case, one might also consider the disturbance sensor dynamics and disturbance sensor noise.

Referring again to Figure 1, notice that the output $\boldsymbol{y}$ is not the output that someone who observes the system would see. Rather, she or he would see a measurement of the output, say $\boldsymbol{y}_m$, which is generated by a physical sensor. This sensor can itself be thought of as a plant which takes $\boldsymbol{y}$ as its input and produces $\boldsymbol{y}_m$ as its output. It can be assumed that the sensor dynamics are stable, in the sense that the output responds in proportion to the input; otherwise the sensor would not be of much use. Because the sensor is a dynamical system, one may expect that it will take some amount of time for it to respond to the signal it is measuring. However, one would hope that the sensor dynamics are much faster than the dynamics of the process being measured; otherwise, the sensor will not be able to "keep up" with $\boldsymbol{y}$ and will give an erroneous measurement. Another possible source of discrepancy between $\boldsymbol{y}$ and $\boldsymbol{y}_m$ is a noise signal $\boldsymbol{v}$. If the measured output is to be compared with the desired output $\boldsymbol{y}_d$ in a feedback loop, then it is important that the controller compensate for any discrepancy between $\boldsymbol{y}$ and $\boldsymbol{y}_m$.

The control design problem is to design a controller (that is, a device or algorithm which generates the control input $\boldsymbol{u}$) in order to make the system, or at least its output, behave in a desired way. Depending on the situation, one might choose the controller to make use of any or all of the following information

- the desired output $\boldsymbol{y}_d$,
- the disturbance measurement $\boldsymbol{w}_m$, and
- the output measurement $\boldsymbol{y}_m$.

Now, assume that every element of the system (the plant, the sensors, and the controller) is linear, time-invariant. In this case, we may exploit the superposition property to express all of the signals as sums of

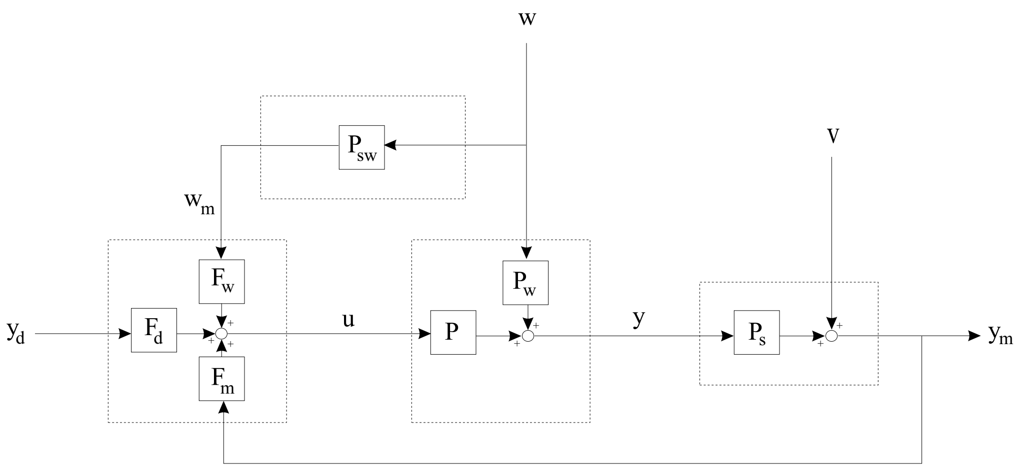

Figure 2: Functional diagram of an LTI control system.

linear operations on other signals. For example, in the Laplace domain, we may write

$$\begin{aligned}
\boldsymbol{u}(s) &= \boldsymbol{F}_d(s)\boldsymbol{y}_d(s) + \boldsymbol{F}_m(s)\boldsymbol{y}_m(s) + \boldsymbol{F}_w(s)\boldsymbol{w}_m(s) \\
&= \boldsymbol{F}_d(s)\boldsymbol{y}_d(s) + \boldsymbol{F}_m(s)\boldsymbol{y}_m(s) + \boldsymbol{F}_w(s)\boldsymbol{P}_{sw}\boldsymbol{w}(s) \\
\boldsymbol{y}(s) &= \boldsymbol{P}(s)\boldsymbol{u}(s) + \boldsymbol{P}_w(s)\boldsymbol{w}(s) \\
\boldsymbol{y}_m(s) &= \boldsymbol{P}_s(s)\boldsymbol{y}(s) + \boldsymbol{v}(s),
\end{aligned}$$

where upper case letters represent transfer function matrices and lower case letters represent signal vectors. Figure 2 shows the system diagram for this more specific case of an LTI system.

Aside: Transfer Functions from LTI State-Space Models. In general, a system of first order linear, time-invariant ODE's may be represented in the state-space form

$$\begin{aligned}
\dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \\
\boldsymbol{y} &= \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u},
\end{aligned}$$

where $\boldsymbol{u}$ is a vector of input signals, $\boldsymbol{y}$ is a vector of output signals, and $\boldsymbol{x}$ is the system state. The set of equations may be transformed into transfer function form by taking the Laplace transform of all of the signals and determining the map from $\boldsymbol{U}(s)$ to $\boldsymbol{Y}(s)$. Doing so, we find that

$$\begin{aligned}
s\boldsymbol{X}(s) - \boldsymbol{x}_0 &= \boldsymbol{A}\boldsymbol{X}(s) + \boldsymbol{B}\boldsymbol{U}(s) \\
\boldsymbol{Y}(s) &= \boldsymbol{C}\boldsymbol{X}(s) + \boldsymbol{D}\boldsymbol{U}(s).
\end{aligned}$$

Rearranging the first equation, we obtain

$$(s\mathbb{I} - \boldsymbol{A})\,\boldsymbol{X}(s) = \boldsymbol{x}_0 + \boldsymbol{B}\boldsymbol{U}(s)$$

or

$$\boldsymbol{X}(s) = (s\mathbb{I} - \boldsymbol{A})^{-1}\,(\boldsymbol{x}_0 + \boldsymbol{B}\boldsymbol{U}(s)),$$

where $\mathbb{I}$ is the identity matrix. Substituting into the output equation gives

$$\boldsymbol{Y}(s) = \boldsymbol{C}\,(s\mathbb{I} - \boldsymbol{A})^{-1}\,(\boldsymbol{x}_0 + \boldsymbol{B}\boldsymbol{U}(s)) + \boldsymbol{D}\boldsymbol{U}(s)$$

or

$$\boldsymbol{Y}(s) = \boldsymbol{C}\,(s\mathbb{I} - \boldsymbol{A})^{-1}\,\boldsymbol{x}_0 + \Big(\boldsymbol{C}\,(s\mathbb{I} - \boldsymbol{A})^{-1}\,\boldsymbol{B} + \boldsymbol{D}\Big)\,\boldsymbol{U}(s).$$

Setting the initial state $\boldsymbol{x}_0$ to zero (as is required to determine a transfer function), we see that the matrix

$$\boldsymbol{P}(s) = \boldsymbol{C}\left(s\mathbb{I} - \boldsymbol{A}\right)^{-1}\boldsymbol{B} + \boldsymbol{D}$$

maps the vector $\boldsymbol{U}(s)$ of input signals to the vector $\boldsymbol{Y}(s)$ of output signals. If the system is "single-input, single output (SISO)," then the matrix transfer function becomes a scalar, say $P(s)$.

The system transfer function represents the complete system dynamics in the form of a map from input signals to output signals. The control problem is to find an input signal that makes the output signal behave in some desired way. If the system being considered is an airplane, then $\boldsymbol{u}(t)$ might represent thrust and elevator inputs and $\boldsymbol{y}(t)$ might represent speed and flight path angle. In designing a longitudinal controller, one would seek thrust and elevator histories which would force the speed and flight path angle to follow some desired time history.

Controller Structures. While Figures 1 and 2 represent a general control system, it is not always necessary or possible to use this complete structure. Following are some simpler control structures which are commonly used in practice:

- *Open-loop:* Neither output nor disturbance measurements are used to compute $\boldsymbol{u}$; the input only depends on the reference signal $\boldsymbol{y}_d$. (In this case, we set $\boldsymbol{F}_m = \boldsymbol{F}_w = \boldsymbol{0}$ and remove the unnecessary output and disturbance sensors.)
- *Closed-loop (1 Degree of Freedom):* The desired output $\boldsymbol{y}_d$ is compared directly with the measured output $\boldsymbol{y}_m$ to generate an error signal which determines $\boldsymbol{u}$. No disturbance measurement is used. (In this case, we set $\boldsymbol{F}_m = -\boldsymbol{F}_d$ and $\boldsymbol{F}_w = \boldsymbol{0}$ and remove the unnecessary disturbance sensor.)
- *Feedforward control:* The disturbance is measured and compensated for by the input $\boldsymbol{u}$. Typically, the problem is one of regulating a set point so that one may take $\boldsymbol{y}_d = \boldsymbol{0}$ without loss of generality. (In this case, we set $\boldsymbol{F}_m = \boldsymbol{F}_d = \boldsymbol{0}$ and remove the unnecessary output sensor.)

There is also a slightly more general form of feedback, called "2 degree of freedom" feedback, in which both $\boldsymbol{y}_d$ and $\boldsymbol{y}_m$ are used to determine $\boldsymbol{u}$, however $\boldsymbol{F}_m \neq -\boldsymbol{F}_d$. Of course, there are other variations, as well. For example, one may combine feedforward and feedback control.

In every case, the goal of the control designer is to make the true output $\boldsymbol{y}$ follow as closely as possible the desired output $\boldsymbol{y}_d$. In other words, the goal is to maintain the error $\boldsymbol{e} = \boldsymbol{y}_d - \boldsymbol{y}$ as small as possible.

Following are some general observations comparing open- and closed-loop control.

Open-loop control . . .

1. . . . can not stabilize an unstable plant.
2. . . . does not attenuate disturbances.
3. . . . does not mitigate sensitivity to plant parameter variations.
4. . . . requires little or no special equipment (such as expensive sensors)

On the other hand, closed-loop control . . .

1. . . . *can* stabilize an unstable plant. (It can also *destabilize* a stable plant!)
2. . . . *does* attenuate disturbances.

3. ... *does* mitigate sensitivity to plant parameter variations.

4. ... *does* require sensors and hardware for processing the sensor signals.

Basic Classical Control Elements. Perhaps the most well-known acronym in the control community is PID. This stands for "proportional-integral-derivative" and it refers to a particular one degree of freedom control structure which is used in a wide variety of applications. As you know, the controller in a one degree of freedom feedback structure generates a control input u to the plant in response to an error signal $e = y_d - y$. The PID controller generates this input as the sum of three signals:

$$u(t) = u_{\mathrm{p}} + u_{\mathrm{i}} + u_{\mathrm{d}}.$$

Proportional action. The first component of the PID control signal is proportional to the error. That is,

$$u_{\mathrm{p}}(t) = k_p e(t),$$

where k_p is the "proportional gain." Large error signals generate large control signals and, as the error decreases, so does the control effort. Of course, the trick is to ensure that large control signals tend to drive the error toward zero instead of making it even bigger. That is, one must check that the proportional control law actually stabilizes the system.

Integral action. The second component of the PID control signal is proportional to the time integral of the error,

$$u_{\mathrm{i}}(t) = k_i \int_0^t e(\tau) d\tau,$$

where k_i is the "integral gain." This term has little effect until the error accumulates sufficiently. Of course, error is a signed quantity, so negative error can cancel with positive error. Consequently, integral action can increase oscillation in the closed-loop system transient response. Again, one must be careful that the integral term does not destabilize the system.

Derivative action. The third component of the PID control signal is proportional to the time derivative of the error,

$$u_{\mathrm{d}}(t) = k_d \frac{de}{dt},$$

where k_d is the "derivative gain." This term responds not to the actual error, but to the rate of increase of error. The derivative term is often described as "anticipatory" because it takes effect before the error ever has a chance to grow.

Summing the various terms gives the classical PID controller:

$$u = k_p e(t) + k_i \int_0^t e(\tau) d\tau + k_d \frac{de}{dt}.$$

In the Laplace domain, we have

$$F(s) = \frac{U(s)}{E(s)} = k_p + k_i \frac{1}{s} + k_d s.$$

Of course, individual components can be removed by setting their respective gain to zero. For example, setting $k_i = 0$ reduces the controller above to a "proportional-derivative" or "PD" controller.

Example: Flight path regulator. For the STOL aircraft we have considered in previous lectures, the transfer function from elevator to flight path angle is

$$P(s) = \frac{\gamma(s)}{\delta e(s)} \approx \frac{7s^3 + 0.1s^2 - 1000s - 8}{s^4 + 4.7s^3 + 8.7s^2 + 0.2s + 0.06}.$$

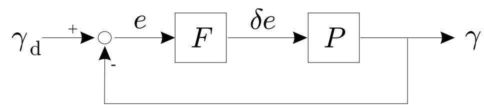

Figure 3: The closed-loop control system for Problem 1.

Suppose we apply proportional feedback

$$F(s) = \frac{\delta e(s)}{e(s)} = k_p$$

in order to regulate the flight path angle to some desired value γ_d. The closed-loop transfer function is

$$\begin{aligned} H(s) = \frac{\gamma(s)}{\gamma_d(s)} &= \frac{P(s)F(s)}{1 + P(s)F(s)} \\ &= \frac{k_p\left(7s^3 + 0.1s^2 - 1000s - 8\right)}{(s^4 + 4.7s^3 + 8.7s^2 + 0.2s + 0.06) + k_p\left(7s^3 + 0.1s^2 - 1000s - 8\right)} \\ &= \frac{7s^3 + 0.1s^2 - 1000s - 8}{s^4 + (4.7 + 7k_p)\, s^3 + (8.7 + 0.1k_p)\, s^2 + (0.2 - 1000k_p)\, s + (0.06 - 8k_p)} \end{aligned}$$

The characteristic values for the closed-loop system are the poles of $H(s)$, i.e., the roots of the denominator polynomial. A necessary condition for these roots to have negative real part is that all coefficients have the same sign. Since the coefficient of s^4 is positive one, we require that all coefficients of lower order terms also be positive. We obtain the following conditions on the proportional gain k_p.

$$k_p > -\frac{4.7}{7}, \qquad k_p > -\frac{8.7}{0.1}, \qquad k_p < \frac{0.2}{1000}, \qquad \text{and} \qquad k_p < \frac{0.06}{8}.$$

Thus, it is necessary that

$$-\frac{4.7}{7} < k_p < \frac{0.2}{1000}.$$

These conditions are only necessary. To obtain necessary and sufficient conditions, one must apply the Routh-Hurwitz stability criterion to be discussed in the next lecture.

References

[1] P. R. Bélanger. *Control Engineering: A Modern Approach.* Saunders College Publishing, Philadelphia, PA, 1995.