Lecture 21: Introduction to Aircraft Control – Pitch Stabilization

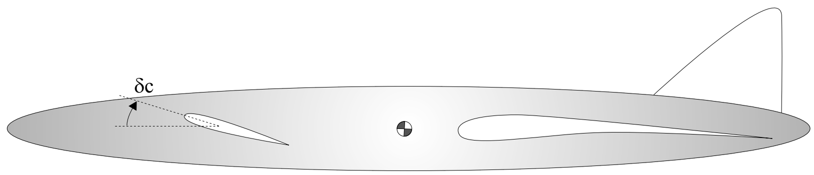

Figure 1: An airplane with canards.

Consider a wind-tunnel model which is pinned to allow pitch rotation about the center of gravity. Rather than a conventional horizontal stabilizer with elevators, the model uses servo-actuated canards to provide longitudinal stability and control. For small angles of attack, the model dynamics are well-described by the second order ODE

$$\ddot{\theta} - M_q\dot{\theta} - M_\alpha\theta = M_{\delta c}\delta c \tag{1}$$

where δc represents the canard deflection. (Since the model is mounted in a wind tunnel, the angle of attack α is identical to the aircraft pitch angle θ.) Pitch damping due to a canard acts to oppose pitch rate just as it does for an aft tail. Thus, one finds that $M_q < 0$. The pitch stiffness, however, is diminished by a forward tail and, for the case shown in Figure 1, one finds that $M_\alpha > 0$. The airplane is not statically stable.

Given a desired pitch angle θ_d, define the error

$$e = \theta_\mathrm{d} - \theta.$$

For simplicity, suppose that $\theta_\mathrm{d} = 0$ and define the following "proportional-derivative (PD)" feedback control law

$$\begin{aligned} \delta c &= k_p e + k_d\dot{e} \\ &= -k_p\theta - k_d\dot{\theta}. \end{aligned}$$

That is, apply a control deflection in which is the sum of terms directly proportional to the error and the rate of increase of the error. Substituting into the dynamic equation gives

$$\ddot{\theta} + (M_{\delta c}k_d - M_q)\dot{\theta} + (M_{\delta c}k_p - M_\alpha)\theta = 0.$$

For stability, we require that

$$M_{\delta c}k_d > M_q \qquad \text{and} \qquad M_{\delta c}k_p > M_\alpha. \tag{2}$$

Since $M_q < 0$ and $M_{\delta c} > 0$, we could actually just choose $k_d = 0$, which would give a simple "proportional feedback" control structure. On the other hand, because $M_\alpha > 0$, we must choose k_p large enough to dominate that destabilizing term. Moreover, in order to obtain an arbitrary closed-loop natural frequency and damping ratio, we should retain the derivative term and choose

$$k_p = \frac{1}{M_{\delta c}}\left(M_\alpha + \omega_n^2\right) \qquad \text{and} \qquad k_d = \frac{1}{M_{\delta c}}\left(M_q + 2\zeta\omega_n\right).$$

Now suppose that θ_d is some nonzero constant. Then we have

$$\ddot{\theta} + (M_{\delta c}k_d - M_q)\dot{\theta} + (M_{\delta c}k_p - M_\alpha)\theta = M_{\delta c}k_p\theta_\mathrm{d}.$$

Assuming conditions (2) hold, one may use the final value theorem (FVT) to show that

$$\lim_{t\to\infty}\theta = \frac{M_{\delta c}k_p}{M_{\delta c}k_p - M_\alpha}\theta_\mathrm{d}$$

If k_p is chosen large enough, then θ will approach θ_{d} in time, however it will never really converge to the desired value. While letting $k_p \to \infty$ would make the error arbitrarily small, there are practical concerns associated with such "high gain" feedback, including actuator limits and destabilization of unmodeled dynamics. A better approach to eliminating the steady-state error is to incorporate an integral term in the controller. At this point, it may be easier to proceed in the s-domain rather than the t-domain

Re-expressing the pitch dynamics (1) in the s-domain, we find that the plant transfer function is

$$P(s) = \frac{\theta(s)}{\delta c(s)} = \frac{M_{\delta c}}{s^2 - M_q s - M_\alpha}.$$

Given the desired pitch angle history $\theta_{\mathrm{d}}(t)$, the error signal is

$$e(s) = \theta_{\mathrm{d}}(s) - \theta(s).$$

In order to stabilize the system, so that the angle of attack may be prescribed as desired, we will implement the PID compensator

$$\begin{aligned} F(s) = \frac{\delta c(s)}{e(s)} &= k_p + k_i \frac{1}{s} + k_d s \\ &= \frac{k_p s + k_i + k_d s^2}{s}. \end{aligned}$$

(The fact that the degree of the numerator polynomial is higher than that of the denominator is a bit problematic, because it suggests that the compensator is acausal, i.e., that current outputs δc depend on future inputs e. In practice, there are simple ways around this problem.)

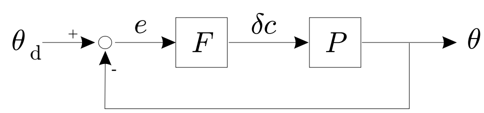

Figure 2: The closed-loop control system.

The feedback control structure is shown in Figure 2. To solve for the closed-loop transfer function (from θ_{d} to θ), we note that

$$\begin{aligned} \theta(s) &= P(s)\delta c(s) \\ &= P(s)(F(s)e(s)) \\ &= P(s)F(s)(\theta_{\mathrm{d}}(s) - \theta(s)). \end{aligned}$$

Solving for θ (and omitting the independent variable s) gives

$$(1 + PF)\theta = PF\theta_{\mathrm{d}}$$

so that the closed-loop transfer function is

$$H(s) = \frac{\theta(s)}{\theta_{\mathrm{d}}(s)} = \frac{PF}{1 + PF}.$$

Substituting the definitions of P and F and manipulating a bit gives

$$\begin{aligned} H(s) &= \frac{\left(\frac{k_p s + k_i + k_d s^2}{s}\right)\left(\frac{M_{\delta c}}{s^2 - M_q s - M_\alpha}\right)}{1 + \left(\frac{k_p s + k_i + k_d s^2}{s}\right)\left(\frac{M_{\delta c}}{s^2 - M_q s - M_\alpha}\right)} \\ &= \frac{\left(k_p s + k_i + k_d s^2\right)\left(M_{\delta c}\right)}{s\left(s^2 - M_q s - M_\alpha\right) + M_{\delta c}\left(k_p s + k_i + k_d s^2\right)} \\ &= \frac{M_{\delta c}\left(k_p s + k_i + k_d s^2\right)}{s^3 + \left(M_{\delta c} k_d - M_q\right) s^2 + \left(M_{\delta c} k_p - M_\alpha\right) s + \left(M_{\delta c} k_i\right)}. \end{aligned}$$

Let's revisit the PD feedback control problem for a moment by assuming that $k_i = 0$. Then the closed-loop transfer function becomes

$$\begin{aligned} H(s) &= \frac{M_{\delta c}\left(k_p s + k_d s^2\right)}{s^3 + \left(M_{\delta c} k_d - M_q\right) s^2 + \left(M_{\delta c} k_p - M_\alpha\right) s} \\ &= \frac{M_{\delta c}\left(k_p + k_d s\right)}{s^2 + \left(M_{\delta c} k_d - M_q\right) s + \left(M_{\delta c} k_p - M_\alpha\right)} \end{aligned} \tag{3}$$

Using the FVT, the steady state response to a step input, say $\bar{\theta}_{\mathrm{d}}\frac{1}{s}$, is

$$\lim_{t\to\infty} \theta = \lim_{s\to 0} sH(s)\left(\bar{\theta}_{\mathrm{d}}\frac{1}{s}\right) = \frac{M_{\delta c} k_p}{M_{\delta c} k_p - M_\alpha}\bar{\theta}_{\mathrm{d}}.$$

Except in the limit $k_p \to \infty$, the steady-state response will not converge to precisely the desired value.

Now consider the PID-controlled system. Stability requires that the roots of the denominator polynomial in (3) have negative real part. (This polynomial plays precisely the same role as the characteristic polynomial for the state matrix in a state-space LTI system.) A necessary condition is that each coefficient in the polynomial be positive, which implies that k_i must be positive (in addition to the previous stability requirements). Suppose stabilizing values of the control gains (k_p, k_i, and k_d) are chosen and consider once again the step response problem:

$$\lim_{t\to\infty} \theta = \lim_{s\to 0} sH(s)\left(\bar{\theta}_{\mathrm{d}}\frac{1}{s}\right) = \frac{M_{\delta c} k_i}{M_{\delta c} k_i}\bar{\theta}_{\mathrm{d}} = \bar{\theta}_{\mathrm{d}}.$$

Thus, integral control eliminates the steady-state error in response to a step input. More generally, integral control is useful for rejecting unmodeled constant biases. So, for example, if the pitch dynamics (1) included a constant term due to some peculiar flow phenomenon, PID control could eliminate that bias, whereas PD control could not.

There are experimentally-based algorithmic techniques for tuning the gains of a PID feedback controller, the most common being the Ziegler-Nichols tuning rules. For more information see [1].

References

[1] K. Ogata. *Modern Control Engineering, Fourth Ed.* Prentice Hall, Upper Saddle River, NJ, 2002.