Lecture 2: Introduction to Static Longitudinal Stability

Transferring moments. Recall that in the previous lecture we began discussing static longitudinal stability. We obtained requirements on the dimensionless *pitch moment coefficient* as a function of the angle of attack α. Specifically, we found that static longitudinal stability requires $C_{m_\alpha} < 0$ and $C_{m_0} > 0$. Before we discuss the various aircraft components and their contributions to C_m, we should review the basic notion of equivalent representations of forces and moments.

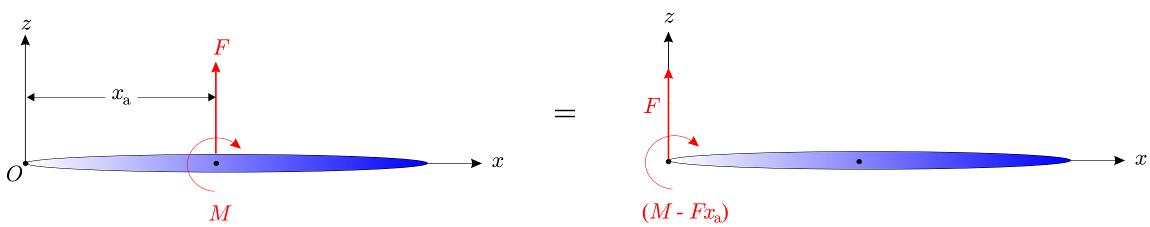

Figure 1: Equivalent force and moment diagrams.

Consider the planar rigid body shown on the left in Figure 1. The body is subject to a force F acting at the point x_a and a moment M, which is a pure couple. For this system, we have

$$\sum F_z = F \qquad \text{and} \qquad \sum M_\mathrm{O} = M - F x_\mathrm{a}.$$

One may easily transfer a set of forces and moments acting at a given point to any other point. For example, one may transfer the force and moment above to the origin O, as shown at the right in Figure 1.

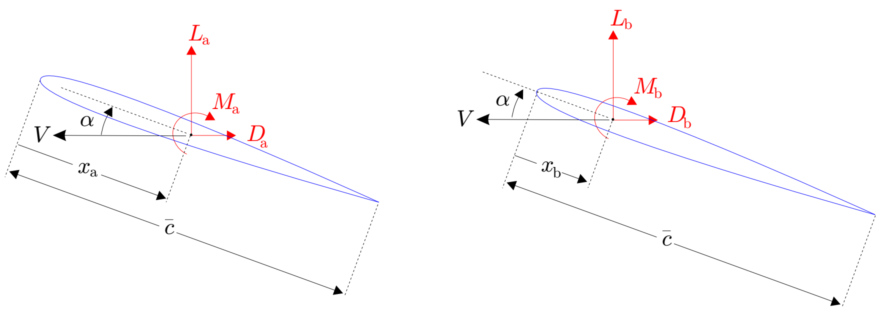

Figure 2: Equivalent force and moment diagram for a wing.

Now consider a rectangular wing. We assume that lift force, drag force, and aerodynamic moment are known, as functions of angle of attack, at the point x_a. (In keeping with aerodynamics convention, the signed distance x is measured positive *aft* from the wing leading edge.)

Suppose we wish to transfer the forces and moment from the point x_a to another point x_b along the chord. The forces are equal at either point:

$$L_\mathrm{b} = L_\mathrm{a} = L \qquad \text{and} \qquad D_\mathrm{b} = D_\mathrm{a} = D.$$

It remains to determine the moment M_b given M_a, L_a, and D_a. First, compute the moment of the system on the left about a particular point, say the leading edge:

$$M_{\mathrm{l.e.}} = M_\mathrm{a} - L(x_\mathrm{a}\cos\alpha) - D(x_\mathrm{a}\sin\alpha).$$

Next, compute the moment of the system on the right about the same point:

$$M_{\mathrm{l.e.}} = M_\mathrm{b} - L(x_\mathrm{b}\cos\alpha) - D(x_\mathrm{b}\sin\alpha).$$

Equating the two expressions for $M_{\text{l.e.}}$ and solving for M_{b} gives

$$M_{\text{b}} = M_{\text{a}} + (L\cos\alpha + D\sin\alpha)(x_{\text{b}} - x_{\text{a}}).$$

Dividing through by $\left(\frac{1}{2}\rho V^2\right) S\bar{c}$ gives

$$C_{m_{\text{b}}} = C_{m_{\text{a}}} + (C_L\cos\alpha + C_D\sin\alpha)\left(\frac{x_{\text{b}}}{\bar{c}} - \frac{x_{\text{a}}}{\bar{c}}\right).$$

(Note: Since we are considering a rectangular wing, the *mean aerodynamic chord* $\bar{c}$ is simply the constant chord length c.) Define the nondimensional distances

$$h_{\text{a}} = \frac{x_{\text{a}}}{\bar{c}} \qquad \text{and} \qquad h_{\text{b}} = \frac{x_{\text{b}}}{\bar{c}}.$$

For small angles of attack[1], we have

$$\begin{aligned} C_{m_{\text{b}}} &= C_{m_{\text{a}}} + (C_L\cos\alpha + C_D\sin\alpha)(h_{\text{b}} - h_{\text{a}}) \\ &\approx C_{m_{\text{a}}} + C_L\left(1 + \frac{C_D}{C_L}\alpha\right)(h_{\text{b}} - h_{\text{a}}). \end{aligned}$$

For a well-designed wing operating below stall, $\frac{C_D}{C_L} \ll 1$ (ignoring, as pathological, the case where $C_L \to 0$). Since we have already assumed that α is small, the product $\frac{C_D}{C_L}\alpha$ may be neglected. We therefore have the following approximate equation for transferring an aerodynamic moment between points on a wing:

$$\boxed{C_{m_{\text{b}}} \approx C_{m_{\text{a}}} + C_L(h_{\text{b}} - h_{\text{a}}).} \tag{1}$$

Aerodynamic reference points. A common reference point for the aerodynamic forces and the pitch moment on a wing is the *aerodynamic center.* The aerodynamic center is that point about which the pitching moment does not vary with angle of attack. To find this point, note that by definition[2]

$$\frac{\partial C_{m_{\text{ac}}}}{\partial\alpha} = 0.$$

Thus, if one knows C_L and $C_{m_{\text{a}}}$, about some point h_{a}, as functions of α (from wind tunnel tests, for example), one may obtain $C_{m_{\text{ac}}}$ through the following procedure:

1. Let h_{b} in equation (1) denote the aerodynamic center.

2. Set the derivative of equation (1) with respect to α equal to zero:

$$0 = \frac{\partial C_{m_{\text{ac}}}}{\partial\alpha} = \frac{\partial C_{m_{\text{a}}}}{\partial\alpha} + \frac{\partial C_L}{\partial\alpha}(h_{\text{ac}} - h_{\text{a}}).$$

   Notice that if C_L and C_m are linear in α (and we will generally assume that they are), all terms in the equation above are constants.

3. Solve for the location of the aerodynamic center:

$$h_{\text{ac}} = h_{\text{a}} - \left(\frac{\partial C_L}{\partial\alpha}\right)^{-1}\frac{\partial C_{m_{\text{a}}}}{\partial\alpha}. \tag{2}$$

[1] The error in this approximation is less than 5% for $|\alpha| \leq \frac{\pi}{12}$ radians $\approx 15°$.

[2] The use of the partial derivative here deserves explanation. Recall that, in general, C_m is considered to be a function of many variables in addition to the angle of attack α. These include pitch rate q, angle of attack rate $\dot{\alpha}$, and others. As used here, the partial indicates that the derivative is taken holding all other quantities constant at their nominal values.

4. Substitute $h_{\rm ac}$ back into equation (1) to obtain $C_{m_{\rm ac}}$.

There is often a simpler approach to find $C_{m_{\rm ac}}$. If $\alpha = \alpha_{0L}$, that is, if the angle of attack corresponds to zero lift, then C_L is zero in equation (1) and the pitch moment (coefficient) is the same everywhere along the wing:

$$C_{m_{\rm b}} = C_{m_{\rm a}} = C_{m_{\rm ac}}.$$

Thus, the (constant) moment about the aerodynamic center has the same value as the zero-lift pitching moment:

$$C_{m_{\rm ac}} = C_{m_{\rm 0L}}.$$

Letting $x_{\rm a} = x_{\rm ac}$, we may re-write the moment transfer formula (1) as

$$\boxed{\begin{aligned} C_{m_b} &\approx C_{m_{\rm ac}} + C_L\left(h_{\rm b} - h_{\rm ac}\right) \\ &= C_{m_{0L}} + C_L\left(h_{\rm b} - h_{\rm ac}\right) \end{aligned}}$$

Aerodynamic data for wings are typically referenced to the wing aerodynamic center or some other wing-related reference point. When writing the equations of motion for an entire aircraft, however, it is most convenient to sum moments about the aircraft center of gravity. Thus, a typical application of the formula above will be to transfer the wing aerodynamic moment to the aircraft center of gravity.

To this point, we have only discussed the aerodynamic center for a rectangular wing. For a more general wing, one introduces the notion of *mean aerodynamic center* $\bar{x}_{\rm ac}$. Appendix C in [1] presents techniques for determining (or approximating) this point, as well as the mean aerodynamic chord $\bar{c}$, for wings of general shape. In subsonic flight, the aerodynamic center is located roughly one-quarter chord aft of the wing's leading edge. In supersonic flight, the aerodynamic center shifts aftward to roughly the half-chord point.

Another reference point which is sometimes important is the point at which the moment generated by the wing vanishes entirely. The *center of pressure* is the point about which the moment due to the aerodynamic force generated by the wing (i.e., the vector sum of lift and drag) precisely balances the pure aerodynamic couple generated by the wing. To find the center of pressure, we solve

$$0 = C_{m_{\rm cp}} = C_{m_{0L}} + C_L\left(h_{\rm cp} - h_{\rm ac}\right)$$

to obtain

$$h_{\rm cp} = h_{\rm ac} - \frac{C_{m_{0L}}}{C_L}.$$

Note that the center of pressure varies with α because C_L varies with α. For this reason, the center of pressure is generally a less useful reference point in aircraft dynamic modeling.

Conditions for Static Longitudinal Stability. Let's return now to the problem of static longitudinal stability. The two requirements we obtained are that the pitch moment coefficient C_m *about the center of gravity* (CG) must have a negative slope and be positive at the zero lift angle of attack α_{0L}. The first condition ensures that a restoring moment is generated in response to small perturbations from $\alpha_{\rm eq}$. The second condition ensures that an equilibrium angle of attack *exists* for which the wing generates the positive lift necessary to balance the airplane's weight.

Given $x_{\rm ac}$ and $C_{m_{\rm ac}}$, we may write

$$C_{m_{\rm cg}} = C_{m_{\rm ac}} + C_L\left(h_{\rm cg} - h_{\rm ac}\right).$$

The first condition for static longitudinal stability is that $C_{m_{\mathrm{cg}_\alpha}} < 0$, where

$$\begin{aligned} C_{m_{\mathrm{cg}_\alpha}} = \frac{\partial C_{m_{\mathrm{cg}}}}{\partial \alpha} &= \frac{\partial C_{m_{\mathrm{ac}}}}{\partial \alpha} + \frac{\partial C_L}{\partial \alpha}\left(h_{\mathrm{cg}} - h_{\mathrm{ac}}\right) \\ &= C_{L_\alpha}\left(h_{\mathrm{cg}} - h_{\mathrm{ac}}\right). \end{aligned}$$

Since $C_{L_\alpha} > 0$ and x is measured positive aft of the leading edge, this condition says that *the center of gravity must be forward of the aerodynamic center.*

Now consider the second condition for static longitudinal stability, that $C_{m_{cg}}|_{\alpha_{0L}} > 0$. This condition ensures that the pitch coefficient curve passes through zero at an angle of attack α_{eq} for which $C_L(\alpha_{\mathrm{eq}})$ is positive. Thus, positive lift will be generated when the pitch moment is zero. Assuming that the dynamic pressure is appropriate, the aircraft's weight will be perfectly balanced by the lift that it generates. We can express this condition as a condition on $C_{m_{\mathrm{ac}}}$ as follows:

$$\begin{aligned} 0 &< \left[C_{m_{\mathrm{cg}}}\right]_{\alpha_{0L}} \\ &= \left[C_{m_{\mathrm{ac}}} + C_L\left(h_{\mathrm{cg}} - h_{\mathrm{ac}}\right)\right]_{\alpha_{0L}} \\ &= C_{m_{\mathrm{ac}}}. \end{aligned}$$

The existence of a statically stable, balanced flight condition requires

$$\boxed{C_{m_{\mathrm{ac}}} = C_{m_{0L}} > 0 \text{ and } C_{m_{\mathrm{cg}_\alpha}} < 0} \quad \text{or, equivalently,} \quad \boxed{C_{m_{\mathrm{ac}}} = C_{m_{0L}} > 0 \text{ and } \left(h_{\mathrm{cg}} - h_{\mathrm{ac}}\right) < 0.}$$

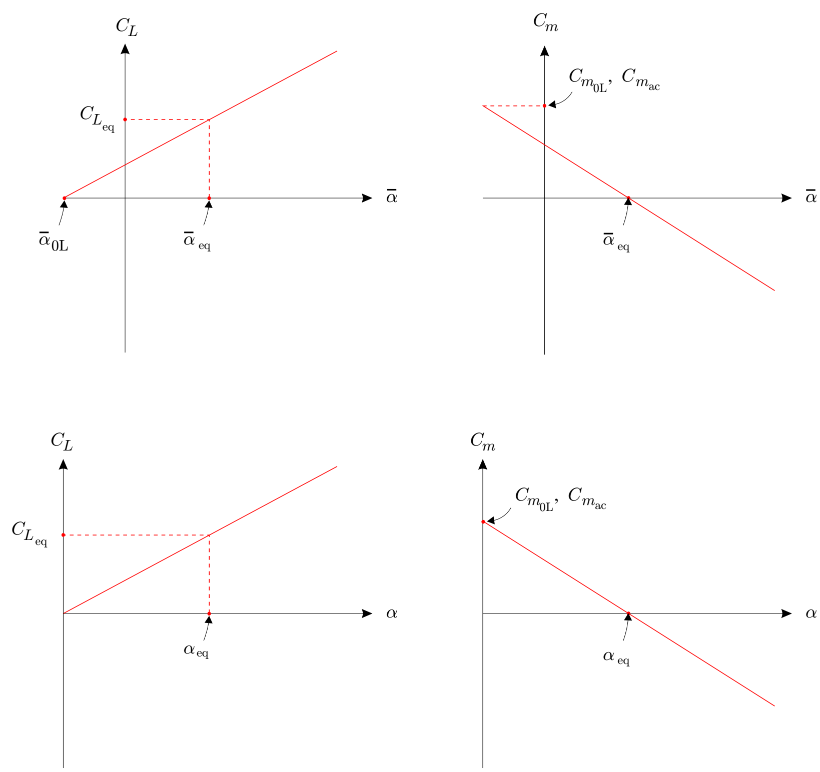

Figure 3: Generic lift and pitch moment coefficient curves. The bottom and top graphs are equivalent except that, in the lower graphs, α is measured from the zero-lift line while, in the upper graphs, $\bar{\alpha}$ is measured from some other fuselage reference line.

Shown in Figure 3 are representative lift and pitch moment coefficient curves, where $C_m = C_{m_{\mathrm{cg}}}$. At the top, C_L and C_m are plotted versus $\bar{\alpha}$ where $\bar{\alpha}$ is *not* measured from the zero-lift line. Below these plots,

C_L and C_m are plotted versus α, which *is* measured from the zero-lift line (meaning $\alpha_{0L} = 0$). Note that the the pitch moment coefficient curve represents a statically stable wing for which the balanced angle of attack corresponds to positive lift. That is, $C_{m_{\rm ac}} = C_{m_{0L}} > 0$ and $C_{m_{{\rm cg}_\alpha}} < 0$.

Example. Suppose lift force and pitch moment data have been obtained for a rectangular "flying wing." Moment data are referred to the one-third chord point $x = \frac{\bar{c}}{3}$ where x is measured aft from the leading edge. The data are given in the table below.

$\bar{\alpha}$ (deg)	C_L	$C_{m_{1/3}}$
0.5	0.2	-0.02
3.0	0.4	0.00
5.5	0.6	0.02
8.0	0.8	0.04

Clearly C_L and $C_{m_{1/3}}$ vary linearly over this range of $\bar{\alpha}$ because each 2.5° increment in $\bar{\alpha}$ gives an equal increment in C_L and $C_{m_{1/3}}$. A least squares fit (though unnecessary for these data!) gives

$$\begin{aligned} C_{L_{\bar{\alpha}}} &= \frac{\partial C_L}{\partial \bar{\alpha}} = 0.08 \text{ deg}^{-1} = 4.6 \text{ rad}^{-1} \\ C_{m_{(1/3)_{\bar{\alpha}}}} &= \frac{\partial C_{m_{1/3}}}{\partial \bar{\alpha}} = 0.008 \text{ deg}^{-1} = 0.46 \text{ rad}^{-1} \end{aligned}$$

We have

$$\begin{aligned} C_L &= C_{L_0} + C_{L_{\bar{\alpha}}} \bar{\alpha} \\ &= 0.16 + 0.08\, \bar{\alpha}_{\rm deg} \\ &= 0.16 + 4.6\, \bar{\alpha}_{\rm rad}. \end{aligned}$$

and

$$\begin{aligned} C_{m_{1/3}} &= C_{m_{(1/3)_0}} + C_{m_{(1/3)_{\bar{\alpha}}}} \bar{\alpha} \\ &= -0.024 + 0.008\, \bar{\alpha}_{\rm deg} \\ &= -0.024 + 0.46\, \bar{\alpha}_{\rm rad}. \end{aligned}$$

Let us compute the aerodynamic center of the wing from these data. Equation (2) gives

$$\begin{aligned} h_{\rm ac} &= h_{\rm a} - \left(\frac{\partial C_L}{\partial \bar{\alpha}}\right)^{-1} \frac{\partial C_{m_{\rm a}}}{\partial \bar{\alpha}} \\ &= \frac{1}{3} - \frac{0.008}{0.08} \\ &\approx 0.23 \end{aligned}$$

which says that the aerodynamic center is at roughly 23% chord. Recall that $C_{m_{\rm ac}} = C_{m_{\rm 0L}}$, and note from the tabulated data that $\bar{\alpha}_{0L} = -2°$. Substituting into the above equation for $C_{m_{1/3}}$ gives

$$C_{m_{\rm ac}} = C_{m_{1/3}}|_{0L} = -0.04$$

Next, we investigate static stability. Notice from the data that the slope of the $C_{m_{1/3}}$ curve is positive. If the CG were located at $x = \frac{c}{3}$, then the flying wing would be statically unstable because small pitch disturbances would drive the state away from the balanced flight condition. The critical CG location at which the slope of the C_m curve becomes zero is the aerodynamic center, which we have computed. Recall that a necessary condition for a flying wing of this design to be statically stable is that the CG be located forward of the aerodynamic center, i.e.,

$$(h_{\text{cg}} - h_{\text{ac}}) < 0,$$

for only then do we have

$$\frac{\partial C_{m_{\text{cg}}}}{\partial \alpha} < 0.$$

This condition alone is not sufficient for static stability, however. The aircraft must also be capable of generating positive lift at the balanced flight condition, in order to balance its weight. Thus, we also require $C_{m_{0L}} > 0$. (See Figure 3.) In our example, $C_{m_{0L}} = C_{m_{ac}} = -0.04$; this flying wing is statically unstable regardless of the CG location. If the CG is forward of the aerodynamic center, so that $C_{m_\alpha} < 0$, then it can not be balanced at a positive angle of attack (i.e., steady, wings-level flight is not and equilibrium). If the CG is aft of the aerodynamic center, there is an equilibrium corresponding to steady, wings-level flight however, because $C_{m_\alpha} > 0$ in this case, the equilibrium is longitudinally unstable. □

References

[1] B. Etkin and L. D. Reid. *Dynamics of Flight: Stability and Control*. John Wiley and Sons, New York, NY, third edition, 1996.