Lecture 18: LTI Systems in State-Space Form

Consider the homogeneous LTI system

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}, \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0. \tag{1}$$

The dynamics of an LTI system are invariant under shifts in the initial time. Without loss of generality, we may assume that $t_0 = 0$ because if t_0 were other than zero, we could simply shift the computed response by that amount of time.

Using Laplace transforms, one may easily compute

$$\begin{aligned} \boldsymbol{X}(s) &= (s\mathbb{I} - \boldsymbol{A})^{-1}\boldsymbol{x}_0 \\ &= \frac{1}{a(s)}\mathrm{Adj}\,(s\mathbb{I} - \boldsymbol{A})\,\boldsymbol{x}_0, \end{aligned}$$

where

$$a(s) = \det(s\mathbb{I} - \boldsymbol{A})$$

and where $\mathrm{Adj}\,(s\mathbb{I} - \boldsymbol{A})$ is the *adjugate matrix*, which is the transpose of the matrix of signed cofactors of $(s\mathbb{I} - \boldsymbol{A})$. The n^{th} order polynomial $a(s)$ is the characteristic polynomial for the matrix $\boldsymbol{A}$; its roots are eigenvalues of $\boldsymbol{A}$. Pursuing the solution further by taking the inverse Laplace transform of each component of $\boldsymbol{X}(s)$, one would find that $\boldsymbol{x}(t)$ is a sum of exponentials whose arguments are the roots of $a(s)$, i.e., the eigenvalues of $\boldsymbol{A}$.

To better understand the nature of solutions to this LTI system, suppose that there exist n linearly independent eigenvectors $\boldsymbol{v}_i$ corresponding to n eigenvalues λ_i (the n roots of $a(\lambda)$).[1]

Proposition: The solution to the homogeneous system (1) with $\boldsymbol{x}_0 = \boldsymbol{v}_i$ is

$$\boldsymbol{v}_i e^{\lambda_i t}.$$

Proof:

$$\frac{d}{dt}\left(\boldsymbol{v}_i e^{\lambda_i t}\right) = \lambda_i \boldsymbol{v}_i e^{\lambda_i t} = \boldsymbol{A}\left(\boldsymbol{v}_i e^{\lambda_i t}\right).$$

□

Suppose that λ_1 and its corresponding eigenvector $\boldsymbol{v}_1$ are real. Then, in the n-dimensional state space, the trajectory either

- follows the direction of the eigenvector $\boldsymbol{v}_1$ in to the origin, if $\lambda_1 < 0$,
- follows the direction of the eigenvector $\boldsymbol{v}_1$ out to infinity, if $\lambda_1 > 0$, or
- remains at $\boldsymbol{v}_1$, if $\lambda_1 = 0$.

In any case, we say that only the "mode" (or "eigenmode") corresponding to λ_1 is excited by the initial state.

By assumption, the eigenvectors $\boldsymbol{v}_i$ of (1) are independent. Therefore, one may express any initial condition $\boldsymbol{x}_0$ in terms of these vectors:

$$\begin{aligned} \boldsymbol{x}_0 &= \sum_{i=1}^{n} c_i \boldsymbol{v}_i \\ &= [\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n]\,\boldsymbol{c} \end{aligned}$$

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[1] If there are repeated eigenvalues, then there may not exist n linearly independent eigenvectors. Ignore this case, for now.

where the coefficients c_i are constant and where $\boldsymbol{c} = [c_1, \cdots, c_n]^T$. Moreover, the set

$$\left\{ \boldsymbol{v}_1 e^{\lambda_1 t}, \cdots, \boldsymbol{v}_n e^{\lambda_n t} \right\}$$

forms a basis of homogeneous solutions from which any solution $\boldsymbol{x}(t)$ to (1) can be formed. Define the matrix

$$\boldsymbol{V}(t) = \left[\boldsymbol{v}_1 e^{\lambda_1 t}, \cdots, \boldsymbol{v}_n e^{\lambda_n t} \right].$$

Then

$$\begin{aligned} \boldsymbol{x}(t) &= c_1 \left(\boldsymbol{v}_1 e^{\lambda_1 t} \right) + \cdots + c_n \left(\boldsymbol{v}_n e^{\lambda_n t} \right) \\ &= \boldsymbol{V}(t)\boldsymbol{c} \\ &= \boldsymbol{V}(t)\boldsymbol{V}(0)^{-1}\boldsymbol{x}_0. \end{aligned}$$

Aside: Define the nonsingular "modal matrix" $\boldsymbol{T} = [\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n]$ and the diagonal matrix $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$. Notice, from the definition of eigenvalues and eigenvectors, that

$$\boldsymbol{A}\boldsymbol{T} = \boldsymbol{T}\boldsymbol{\Lambda}.$$

Using $\boldsymbol{T}$, one may transform the state equations (1) into *modal coordinates*

$$\boldsymbol{z} = \boldsymbol{T}^{-1}\boldsymbol{x}.$$

The transformed state equations are

$$\dot{\boldsymbol{z}} = \boldsymbol{\Lambda}\boldsymbol{z}, \qquad \boldsymbol{z}(t_0) = \boldsymbol{T}^{-1}\boldsymbol{x}_0.$$

Notice that these are simply n *decoupled*, first order, LTI equations. □

So far, we have ignored the question of what happens when λ_i is complex. Suppose that $\lambda_1 = \alpha + i\beta$ and that $\lambda_2 = \bar{\lambda}_1 = \alpha - i\beta$. It is easily proved that $\boldsymbol{v}_1$ must be complex, say $\boldsymbol{v}_1 = \boldsymbol{a} + i\boldsymbol{b}$, and that $\boldsymbol{v}_2 = \bar{\boldsymbol{v}}_1 = \boldsymbol{a} - i\boldsymbol{b}$ is its complex conjugate.

We previously considered what happens when the initial state is an eigenvector. Obviously we can not have a complex initial state, so here we consider what happens when the initial state is either $\boldsymbol{a}$ or $\boldsymbol{b}$, both of which are real vectors. First, suppose the initial state is

$$\boldsymbol{x}_0 = \boldsymbol{a} = \frac{1}{2}(\boldsymbol{v}_1 + \boldsymbol{v}_2) = \boldsymbol{V}(0) \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Then, from our previous discussion we have

$$\begin{aligned} \boldsymbol{x}(t) = \boldsymbol{V}(t)\boldsymbol{V}(0)^{-1}\boldsymbol{x}_0 = \boldsymbol{V}(t) \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} &= \frac{1}{2} \left((\boldsymbol{a} + i\boldsymbol{b}) e^{(\alpha + i\beta)t} + (\boldsymbol{a} - i\boldsymbol{b}) e^{(\alpha - i\beta)t} \right) \\ &= \frac{1}{2} e^{\alpha t} \left((\boldsymbol{a} + i\boldsymbol{b}) e^{i\beta t} + (\boldsymbol{a} - i\boldsymbol{b}) e^{-i\beta t} \right) \\ &= e^{\alpha t} \left(\boldsymbol{a} \frac{1}{2} \left(e^{i\beta t} + e^{-i\beta t} \right) - \boldsymbol{b} \frac{1}{2i} \left(e^{i\beta t} - e^{-i\beta t} \right) \right) \\ &= e^{\alpha t} \left(\boldsymbol{a} \cos \beta t - \boldsymbol{b} \sin \beta t \right), \end{aligned}$$

which is real-valued, as $\boldsymbol{x}(t)$ must be.

Similarly, one may show that the zero-input response to the initial condition $\boldsymbol{x}_0 = \boldsymbol{b}$ is

$$\boldsymbol{x}(t) = e^{\alpha t}\left(\boldsymbol{a}\sin\beta t + \boldsymbol{b}\cos\beta t\right).$$

An interesting observation is that

$$\boldsymbol{A}[\boldsymbol{a},\boldsymbol{b}] = [\boldsymbol{a},\boldsymbol{b}]\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.$$

While it is also true that

$$\boldsymbol{A}[\boldsymbol{v}_1,\boldsymbol{v}_2] = [\boldsymbol{v}_1,\boldsymbol{v}_2]\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

the former expression only involves real-valued elements and may therefore be more convenient to work with. Suppose we use this observation to define a different decomposition $\boldsymbol{A} = \tilde{\boldsymbol{T}}\tilde{\boldsymbol{\Lambda}}\tilde{\boldsymbol{T}}^{-1}$ where the columns of $\tilde{\boldsymbol{T}}$ are either

- the real eigenvectors $\boldsymbol{v}_i$, corresponding to real eigenvalues λ_i, or
- the real and imaginary component vectors ("$\boldsymbol{a}$" and "$\boldsymbol{b}$") of complex conjugate pairs of eigenvectors corresponding to complex conjugate pairs of eigenvalues.

For example, suppose that the λ_1 and λ_2 are complex conjugates while the remaining $n-2$ eigenvalues are real, with corresponding real, independent eigenvectors $\boldsymbol{v}_i$. Then defining the real-valued transformation matrix

$$\tilde{\boldsymbol{T}} = [\boldsymbol{a},\boldsymbol{b},\boldsymbol{v}_3,\ldots,\boldsymbol{v}_n]$$

gives the the state equations written in the "quasi-modal coordinates" $\tilde{\boldsymbol{z}} = \tilde{\boldsymbol{T}}^{-1}\boldsymbol{x}$:

$$\begin{pmatrix} \begin{pmatrix} \dot{\tilde{z}}_1 \\ \dot{\tilde{z}}_2 \end{pmatrix} \\ \dot{\tilde{z}}_3 \\ \vdots \\ \dot{\tilde{z}}_n \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} & \boldsymbol{0} \\ \boldsymbol{0} & \begin{pmatrix} \lambda_3 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix} \\ \tilde{z}_3 \\ \vdots \\ \tilde{z}_n \end{pmatrix}.$$

The system response is

$$\begin{pmatrix} \begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix} \\ \tilde{z}_3 \\ \vdots \\ \tilde{z}_n \end{pmatrix} = \begin{pmatrix} e^{\alpha t}\begin{pmatrix} \cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t \end{pmatrix} & \boldsymbol{0} \\ \boldsymbol{0} & \begin{pmatrix} e^{\lambda_3 t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n t} \end{pmatrix} \end{pmatrix} \tilde{\boldsymbol{z}}_0.$$

When there are *repeated* real or complex conjugate eigenvalues, things can become slightly more complicated. In this case, there may not be n linearly independent eigenvalues, meaning one cannot construct the nonsingular modal matrix $\boldsymbol{T}$. However, the ideas discussed here can be extended using a device known as "Jordan form."

To better understand the role of eigenvalues and eigenvectors in the initial condition response of a linear, time-invariant system of equations, it is helpful to consider the special case of planar systems. Each of the following examples involves the LTI system

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^2$.

Example 1.

$$\boldsymbol{A} = \begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix}$$

By definition, an eigenvalue λ of a matrix $\boldsymbol{A}$ is a (possibly complex) scalar for which there exists a nonzero vector $\boldsymbol{v}$, called an eigenvector, satisfying $\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v}$. Since $(\lambda \mathbb{I} - \boldsymbol{A})\boldsymbol{v} = \boldsymbol{0}$ and $\boldsymbol{v}$ is nonzero, the matrix $\lambda \mathbb{I} - \boldsymbol{A}$ has a nontrivial null space which contains the vector $\boldsymbol{v}$. It follows that $\lambda \mathbb{I} - \boldsymbol{A}$ is not full rank and its determinant is zero. The eigenvalues of $\boldsymbol{A}$ are therefore the scalar values λ for which $\det(\lambda \mathbb{I} - \boldsymbol{A}) = 0$; that is, the eigenvalues are the roots of

$$\begin{aligned} \det(\lambda \mathbb{I} - \boldsymbol{A}) = \det \begin{pmatrix} \lambda - 1 & 3 \\ 3 & \lambda - 1 \end{pmatrix} &= (\lambda - 1)^2 - 9 \\ &= \lambda^2 - 2\lambda - 8 \\ &= (\lambda - 4)(\lambda + 2). \end{aligned}$$

The eigenvalues are thus -2 and 4. To compute the eigenvectors, we must determine the null space of $\lambda \mathbb{I} - \boldsymbol{A}$ for each of the two eigenvalues. First, we must find $\boldsymbol{v}_1$ such that

$$\boldsymbol{0} = (-2\mathbb{I} - \boldsymbol{A})\boldsymbol{v}_1 = \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} \boldsymbol{v}_1.$$

Of course, the equations are redundant because the matrix is rank deficient. That is precisely how an eigenvalue is defined! One choice of eigenvector (there are infinitely many choices) is obtained by setting the first component $\boldsymbol{v}_{1_1} = 1$ and computing the second, which gives $\boldsymbol{v}_{1_2} = 1$ or

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Next, we must find $\boldsymbol{v}_2$ such that

$$\boldsymbol{0} = (4\mathbb{I} - \boldsymbol{A})\boldsymbol{v}_2 = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \boldsymbol{v}_2.$$

Proceeding as before, we find

$$\boldsymbol{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Forming the transformation matrix

$$\boldsymbol{T} = [\boldsymbol{v}_1,\ \boldsymbol{v}_2] = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and defining

$$\boldsymbol{z} = \boldsymbol{T}^{-1}\boldsymbol{x},$$

we obtain the equations of motion in modal coordinates

$$\dot{\boldsymbol{z}} = \boldsymbol{\Lambda}\boldsymbol{z} = \begin{pmatrix} -2 & 0 \\ 0 & 4 \end{pmatrix} \boldsymbol{z}.$$

If $\boldsymbol{z}_0 = [1, 0]^T$, which corresponds to $\boldsymbol{x}_0 = \boldsymbol{v}_1$, then only the first mode is excited and the response is

$$\boldsymbol{z}(t) = \begin{pmatrix} e^{-2t} \\ 0 \end{pmatrix} \quad \text{or} \quad \boldsymbol{x}(t) = \boldsymbol{T}\boldsymbol{z}(t) = \boldsymbol{T}\left[e^{-2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} = \boldsymbol{v}_1 e^{\lambda_1 t}.$$

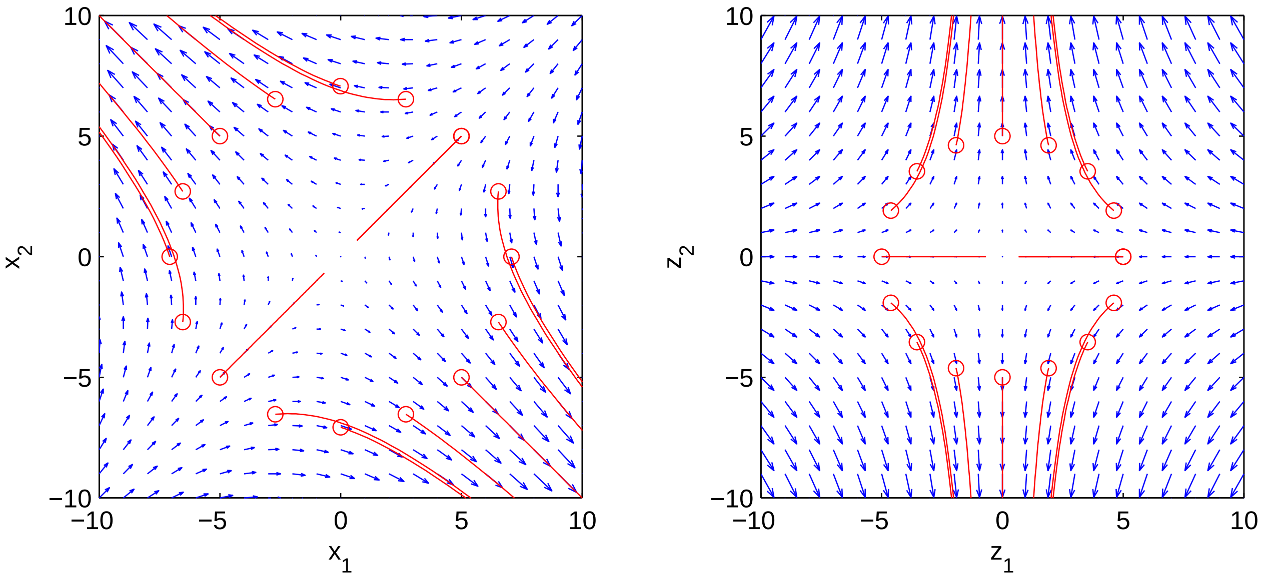

Figure 1: Phase portrait for Example #1. Original coordinates (left) and quasi-modal coordinates (right).

If $\boldsymbol{z}_0 = [0,1]^T$, which corresponds to $\boldsymbol{x}_0 = \boldsymbol{v}_2$, then only the second mode is excited and the response is

$$\boldsymbol{z}(t) = \begin{pmatrix} 0 \\ e^{4t} \end{pmatrix} \quad \text{or} \quad \boldsymbol{x}(t) = \boldsymbol{T}\boldsymbol{z}(t) = \boldsymbol{T}\left[e^{4t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t} = \boldsymbol{v}_2 e^{\lambda_2 t}.$$

More generally, the response will be a linear combination of these two responses as depicted in the subplot to the right in Figure 1. The plot shows the system response, in modal coordinates, to several initial conditions (marked by small circles). These initial condition responses are superimposed on a plot of the "vector field" $\boldsymbol{\Lambda z}$. (In dynamical systems theory, a vector field defines, at any point $\boldsymbol{z}$, a velocity vector $\dot{\boldsymbol{z}}$.) We may convert the modal response back into $\boldsymbol{x}$ coordinates (shown in the subplot to the left) by multiplying $\boldsymbol{z}(t)$ by the transformation matrix $\boldsymbol{T}$. Notice that the line $z_2 = 0$, corresponding to a pure "Mode 1" response transforms to the line $x_2 = x_1$, which is defined by $\alpha \boldsymbol{v}_1$ for all $\alpha \in \mathbb{R}$. Similarly, the line $z_1 = 0$ transforms to the line $x_2 = -x_1$, which corresponds to the eigenvector $\boldsymbol{v}_2$.

ASIDE: For a planar system with two real eigenvalues, the equilibrium at the origin is called a *stable node* or a *sink* if both eigenvalues are real and negative. If the eigenvalues are both real and positive, the equilibrium is called an *unstable node* or a *source*. If one eigenvalue is positive and the other is negative, as in the example above, then the equilibrium is called a *saddle node*. □

Example 2.

$$\boldsymbol{A} = \begin{pmatrix} -1 & \frac{1}{2} \\ -2 & -1 \end{pmatrix}$$

The eigenvalues are

$$\lambda_{1,2} = -1 \pm i$$

and the eigenvectors are

$$\boldsymbol{v}_{1,2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \pm i \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

Define

$$\boldsymbol{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

Using the transformation matrix $\tilde{\boldsymbol{T}} = [\boldsymbol{a},\ \boldsymbol{b}]$, define

$$\tilde{\boldsymbol{z}} = \tilde{\boldsymbol{T}}^{-1}\boldsymbol{x}.$$

The equations of motion in quasi-modal coordinates are

$$\dot{\tilde{\boldsymbol{z}}} = \tilde{\boldsymbol{\Lambda}}\tilde{\boldsymbol{z}} = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \boldsymbol{z}.$$

The general response is

$$\tilde{\boldsymbol{z}}(t) = e^{-t} \begin{pmatrix} \cos(\beta t) & \sin(\beta t) \\ -\sin(\beta t) & \cos(\beta t) \end{pmatrix} \tilde{\boldsymbol{z}}_0.$$

If $\tilde{\boldsymbol{z}}_0 = [1, 0]^T$, which corresponds to $\boldsymbol{x}_0 = \boldsymbol{a}$, then the response is

$$\tilde{\boldsymbol{z}}(t) = e^{-t} \begin{pmatrix} \cos \beta t \\ -\sin \beta t \end{pmatrix}.$$

If $\tilde{\boldsymbol{z}}_0 = [0, 1]^T$, which corresponds to $\boldsymbol{x}_0 = \boldsymbol{b}$, then the response is

$$\tilde{\boldsymbol{z}}(t) = e^{-t} \begin{pmatrix} \sin \beta t \\ \cos \beta t \end{pmatrix}.$$

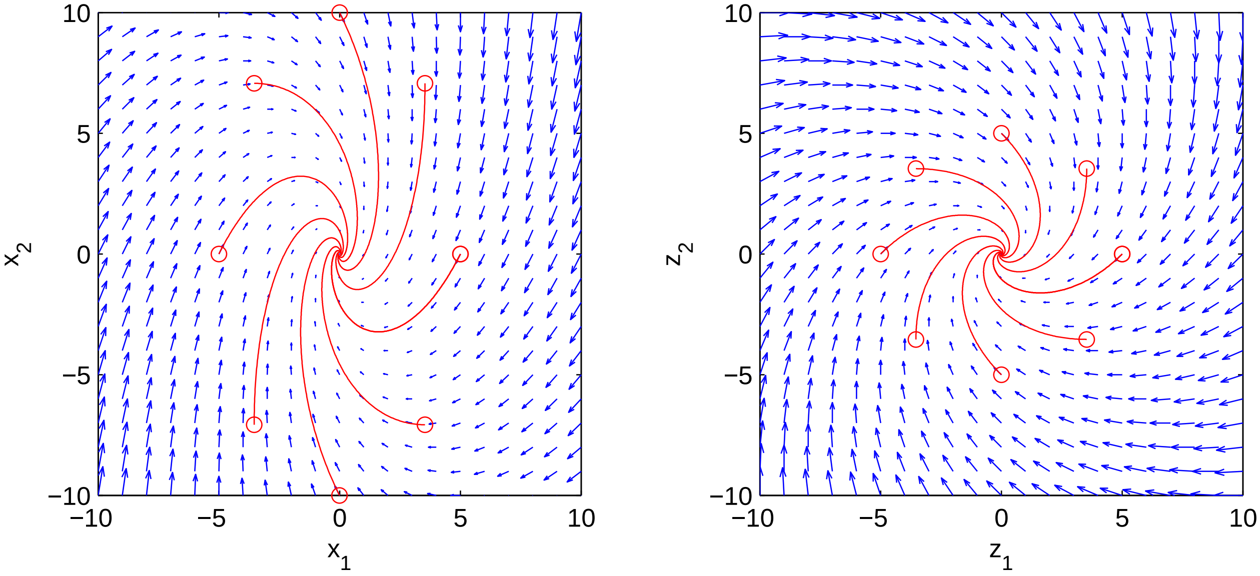

Figure 2: Phase portrait for Example #2. Original coordinates (left) and quasi-modal coordinates (right).

More generally, the response will be a linear combination of these two responses. The subplot to the right in Figure 2 shows the system response to several initial conditions superimposed on a plot of the vector field $\tilde{\boldsymbol{\Lambda}}\tilde{\boldsymbol{z}}$. We may convert this response back into $\boldsymbol{x}$ coordinates (shown to the left) by multiplying $\tilde{\boldsymbol{z}}(t)$ by the transformation matrix $\tilde{\boldsymbol{T}}$.

ASIDE: For a planar system with complex conjugate eigenvalues, as above, the equilibrium at the origin is called a *stable focus* if $\alpha < 0$. If $\alpha > 0$, then the equilibrium is called an *unstable focus*. If $\alpha = 0$, then the equilibrium is called a *center*. □