Lecture 17: Lateral-Directional Stability Derivatives

The v-derivatives. Equivalently, the β-derivatives: Y_β, L_β, and N_β. We have assumed that the asymmetric force Y and moments L and N depend only on the asymmetric state and control variables v, p, r, δa, and δr. We first consider the dependence of Y, L, and N on v. Equivalently, we may consider their dependence on β. Recall that

$$\sin\beta = \frac{v}{V}.$$

Differentiating with respect to v gives

$$\cos\beta\frac{\partial\beta}{\partial v} = \frac{1}{V} - \frac{v^2}{V^3} = \frac{1}{V}\left(1-\sin^2\beta\right) = \frac{1}{V}\cos^2\beta.$$

For small perturbations from nominal flight, we have

$$\frac{\partial\beta}{\partial v} \approx \frac{1}{u_0}$$

Thus, β and v are directly proportional when operating near the nominal flight condition. For the linearized equations at least, we may write

$$\frac{\partial}{\partial\beta} = u_0\frac{\partial}{\partial v}.$$

Stability derivatives may appear either with respect to β or with respect to v; for small perturbations, the two are directly proportional. In particular, we have

$$Y_\beta = u_0 Y_v, \qquad L_\beta = u_0 L_v, \qquad \text{and} \qquad N_\beta = u_0 N_v.$$

Consider first the term C_{Y_β}. The primary contributor to this term is the vertical tail, for which we may write

$$Y_{\text{vt}} = C_{L_{\alpha_{\text{vt}}}}\left(-\beta+\sigma(\beta)\right)\left(\frac{1}{2}\rho V^2\right)S_{\text{vt}}.$$

(Recall that any change in tail "efficiency" due to differences in dynamic pressure at the tail and in the free stream is assumed to be represented in the lift curve slope $C_{L_{\alpha_{\text{vt}}}}$.) Normalizing by $\left(\frac{1}{2}\rho V^2\right)S$ and differentiating with respect to β gives

$$C_{Y_\beta} = -C_{L_{\alpha_{\text{vt}}}}\left(1-\frac{d\sigma}{d\beta}\right)\frac{S_{\text{vt}}}{S}.$$

The term C_{n_β} was discussed in a previous lecture. It depends on the wing-body and the vertical tail:

$$C_{n_\beta} = C_{n_{\beta_{\text{wb}}}} + V\!\!\!\!-_{\text{V}} C_{L_{\alpha_{\text{vt}}}}\left(1-\frac{d\sigma}{d\beta}\right).$$

The term $V\!\!\!\!-_{\text{V}}$ above is the vertical tail volume ratio.

As discussed in Lecture 10, the term C_{l_β} owes primarily to the dihedral effect. Having computed the nondimensional stability coefficients C_{Y_β}, C_{l_β}, and C_{n_β}, the dimensional stability derivatives are

$$Y_\beta = C_{Y_\beta}\left(\frac{1}{2}\rho u_0^2\right)S, \qquad L_\beta = C_{l_\beta}\left(\frac{1}{2}\rho u_0^2\right)Sb, \qquad \text{and} \qquad N_\beta = C_{n_\beta}\left(\frac{1}{2}\rho u_0^2\right)Sb.$$

The p-derivatives: Y_p, L_p, and N_p. When an airplane in equilibrium flight experiences a perturbation which results in a nonzero roll rate Δp, the result is a change in lift generated by each of the lifting

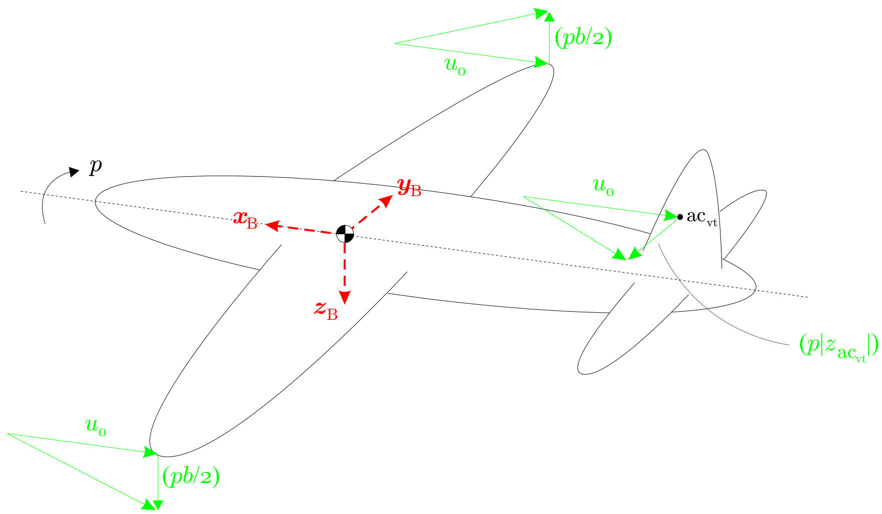

Figure 1: Effect of roll rate.

surfaces. The wing, for example, experiences a change in lift profile such that the downward moving wing experiences an increase in lift and the upward moving wing experiences a decrease in lift. (The increment in lift varies linearly from wingtip to wingtip.) The same is true for the horizontal tail. Also, an increment in angle of attack is induced at the vertical tail. All of these effects contribute to the three roll rate stability derivatives

$$C_{Y_p} = \frac{\partial C_Y}{\partial \hat{p}}, \qquad C_{l_p} = \frac{\partial C_l}{\partial \hat{p}}, \qquad C_{n_p} = \frac{\partial C_n}{\partial \hat{p}} \qquad \text{where} \qquad \hat{p} = \frac{p}{\frac{u_0}{b/2}},$$

or

$$C_{Y_p} = \frac{2u_0}{b}\frac{\partial C_Y}{\partial p}, \qquad C_{l_p} = \frac{2u_0}{b}\frac{\partial C_l}{\partial p}, \qquad C_{n_p} = \frac{2u_0}{b}\frac{\partial C_n}{\partial p}.$$

Etkin and Reid [1] suggest that the side force due to roll rate can often be neglected. Primary contributors to C_{Y_p} are the vertical tail and the wing. The vertical tail contribution can be estimated by recognizing that the non-zero roll rate induces a sideslip angle

$$\alpha_{\rm vt} = \frac{-p z_{\rm ac_{vt}}}{u_0}$$

at the vertical tail. (While the sideslip angle actually varies over the span of the vertical tail, we assume that the variations cancel out so that the value above is a good approximation.) For positive roll rate $p > 0$ and an above-board vertical tail, the sideslip angle $\alpha_{\rm vt}$ is positive. Consequently, the vertical tail exerts

- a side force in the negative $\boldsymbol{y}_{\rm B}$ direction,
- a negative roll moment, which opposes the roll rate, and
- a positive yaw moment.

While the wing contribution is not easily estimated, it is fairly easy to understand physically. Because the downward moving wing generates more lift and the upward moving wing generates less, the primary effect is a roll moment which opposes the roll rate. Thus C_{l_p} is often called the "roll damping" derivative. Also, because drag is proportional to the square of lift, the downward moving wing experiences greater drag resulting in a yaw moment which is positive for positive roll rate.

Techniques for estimating C_{l_p} and C_{n_p} are given in Appendix B.10 of [1]. Having computed the nondimensional stability coefficients C_{Y_p}, C_{l_p}, and C_{n_p}, the dimensional stability derivatives are

$$Y_p = C_{Y_p}\left(\frac{b}{2u_0}\right)\left(\frac{1}{2}\rho u_0^2\right)S, \qquad L_p = C_{l_p}\left(\frac{b}{2u_0}\right)\left(\frac{1}{2}\rho u_0^2\right)Sb, \qquad \text{and} \qquad N_p = C_{n_p}\left(\frac{b}{2u_0}\right)\left(\frac{1}{2}\rho u_0^2\right)Sb.$$

The r-derivatives: Y_r, L_r, and N_r. The asymmetric force and moments vary with yaw rate through two primary effects. First, and more obviously, the vertical tail experiences a change in angle of attack due to the yaw rate; a positive yaw rate results in a negative sideslip angle, as shown in Figure 2. The effect of the resulting side force generated by the tail is easily estimated. Second, the lift generated by the wings varies due to the relative change in forward speed at different points along the span. This contribution is not as easy to estimate, though it is fairly easy to understand. For a positive yaw rate, the airspeed of the left wing increases and the airspeed of the right wing decreases. The result is an increase in lift and drag on the left wing and a decrease in lift and drag on the right. These changes result in a negative yaw moment, which opposes the positive yaw rate, and a positive roll moment. All of these effects contribute to the three yaw rate stability derivatives

$$C_{Y_r} = \frac{\partial C_Y}{\partial \hat{r}}, \qquad C_{l_r} = \frac{\partial C_l}{\partial \hat{r}}, \qquad C_{n_r} = \frac{\partial C_n}{\partial \hat{r}} \qquad \text{where} \qquad \hat{r} = \frac{r}{\frac{u_0}{b/2}},$$

or

$$C_{Y_r} = \frac{2u_0}{b}\frac{\partial C_Y}{\partial r}, \qquad C_{l_r} = \frac{2u_0}{b}\frac{\partial C_l}{\partial r}, \qquad C_{n_r} = \frac{2u_0}{b}\frac{\partial C_n}{\partial r}.$$

To estimate the effect of the vertical tail on the stability derivatives C_{Y_r}, C_{l_r}, and C_{n_r}, note that a side force Y results:

$$Y = -C_{L_{\alpha_{\text{vt}}}}\Delta\beta\left(\frac{1}{2}\rho V^2\right)S_{\text{vt}},$$

where

$$\Delta\beta = -\frac{r l_{\text{vt}}}{u_0}.$$

Normalizing by $\left(\frac{1}{2}\rho V^2\right)S$ gives

$$C_Y = -C_{L_{\alpha_{\text{vt}}}}\left(-\frac{r l_{\text{vt}}}{u_0}\right)\frac{S_{\text{vt}}}{S}$$

Differentiating with respect to r and re-normalizing gives

$$\begin{aligned} C_{Y_r} &= \left(\frac{2u_0}{b}\right)C_{L_{\alpha_{\text{vt}}}}\left(\frac{l_{\text{vt}}}{u_0}\right)\frac{S_{\text{vt}}}{S} \\ &= 2C_{L_{\alpha_{\text{vt}}}}\mathcal{V}_{\text{v}} \end{aligned}$$

The side force Y resulting from a non-zero yaw rate generates a yaw moment

$$\begin{aligned} N &= -Y l_{\text{vt}} \\ &= C_{L_{\alpha_{\text{vt}}}}\Delta\beta\left(\frac{1}{2}\rho V^2\right)S_{\text{vt}}l_{\text{vt}} \\ &= C_{L_{\alpha_{\text{vt}}}}\left(-\frac{r l_{\text{vt}}}{u_0}\right)\left(\frac{1}{2}\rho V^2\right)S_{\text{vt}}l_{\text{vt}}. \end{aligned}$$

Note that the tail contribution to yaw moment is negative for a positive yaw rate. The yaw moment due to the tail thus opposes yaw rate; the effect is referred to as "yaw rate damping." Normalizing by $\left(\frac{1}{2}\rho V^2\right)Sb$ gives

$$C_n = C_{L_{\alpha_{\text{vt}}}}\left(-\frac{r l_{\text{vt}}}{u_0}\right)\mathcal{V}_{\text{v}}$$

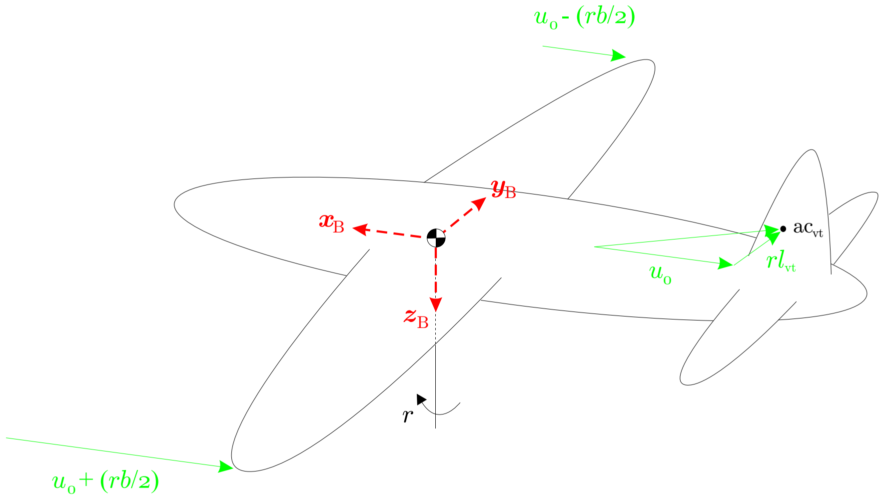

Figure 2: Effect of roll rate.

Differentiating with respect to r and re-normalizing gives

$$\begin{aligned} C_{n_r} &= \left(\frac{2u_0}{b}\right) C_{L_{\alpha_{\mathrm{vt}}}} \left(-\frac{l_{\mathrm{vt}}}{u_0}\right) \mathcal{V}_{\mathrm{v}} \\ &= -2C_{L_{\alpha_{\mathrm{vt}}}} \mathcal{V}_{\mathrm{v}} \frac{l_{\mathrm{vt}}}{b}. \end{aligned}$$

The vertical tail's contribution to roll moment arises because the tail aerodynamic center is a signed distance $z_{\mathrm{ac}_{\mathrm{vt}}}$ below the $\boldsymbol{x}_{\mathrm{B}}$ axis. (The distance $z_{\mathrm{ac}_{\mathrm{vt}}}$ is negative for a top-mounted vertical tail.) The tail contribution to the two moment coefficients C_{n_r} and C_{l_r} are directly proportional; neglecting other contributions, we have

$$\begin{aligned} C_{l_r} &= \frac{z_{\mathrm{ac}_{\mathrm{vt}}}}{l_{\mathrm{vt}}} C_{n_r} \\ &= -\frac{z_{\mathrm{ac}_{\mathrm{vt}}}}{l_{\mathrm{vt}}} \left(2C_{L_{\alpha_{\mathrm{vt}}}} \mathcal{V}_{\mathrm{v}} \frac{l_{\mathrm{vt}}}{b}\right). \end{aligned}$$

Note that C_{l_r} is positive for a top-mounted vertical tail. This means that the tail will provide a positive roll moment in response to a positive yaw rate. The roll moment contribution due to the vertical tail acts in the same direction as that of the wing.

Having computed the nondimensional stability coefficients C_{Y_r}, C_{l_r}, and C_{n_r}, the dimensional stability derivatives are

$$Y_r = C_{Y_r} \left(\frac{b}{2u_0}\right) \left(\frac{1}{2}\rho u_0^2\right) S, \qquad L_r = C_{l_r} \left(\frac{b}{2u_0}\right) \left(\frac{1}{2}\rho u_0^2\right) Sb, \qquad \text{and} \qquad N_r = C_{n_r} \left(\frac{b}{2u_0}\right) \left(\frac{1}{2}\rho u_0^2\right) Sb.$$

References

[1] B. Etkin and L. D. Reid. *Dynamics of Flight: Stability and Control.* John Wiley and Sons, New York, NY, third edition, 1996.