Lecture 15: Longitudinal and Lateral-Directional Dynamics

Recall, from the last lecture, that the perturbation equations for a rigid aircraft linearized about wings-level equilibrium flight are

$$\begin{aligned}
\Delta\dot{x} &= \cos\theta_0 \Delta u - (u_0 \sin\theta_0)\Delta\theta + \sin\theta_0 \Delta w \\
\Delta\dot{y} &= (u_0 \cos\theta_0)\Delta\psi + \Delta v \\
\Delta\dot{z} &= -\sin\theta_0 \Delta u - (u_0 \cos\theta_0)\Delta\theta + \cos\theta_0 \Delta w \\
\Delta\dot{\phi} &= \Delta p + \tan\theta_0 \Delta r \\
\Delta\dot{\theta} &= \Delta q \\
\Delta\dot{\psi} &= \sec\theta_0 \Delta r \\
m\Delta\dot{u} &= (X_u \Delta u + X_w \Delta w + X_{\delta e}\Delta\delta e + X_{\delta T}\Delta\delta T) - mg\cos\theta_0 \Delta\theta \\
m\Delta\dot{v} &= -mu_0 \Delta r + (Y_v \Delta v + Y_p \Delta p + Y_r \Delta r + Y_{\delta a}\Delta\delta a + Y_{\delta r}\Delta\delta r) + mg\cos\theta_0 \Delta\phi \\
m\Delta\dot{w} &= mu_0 \Delta q + (Z_u \Delta u + Z_w \Delta w + Z_{\dot{w}}\Delta\dot{w} + Z_q \Delta q + Z_{\delta e}\Delta\delta e + Z_{\delta T}\Delta\delta T) - mg\sin\theta_0 \Delta\theta \\
I_{\mathrm{x}}\Delta\dot{p} - I_{\mathrm{xz}}\Delta\dot{r} &= (L_v \Delta v + L_p \Delta p + L_r \Delta r + L_{\delta a}\Delta\delta a + L_{\delta r}\Delta\delta r) \\
I_{\mathrm{y}}\Delta\dot{q} &= (M_u \Delta u + M_w \Delta w + M_{\dot{w}}\Delta\dot{w} + M_q \Delta q + M_{\delta e}\Delta\delta e + M_{\delta T}\Delta\delta T) \\
I_{\mathrm{z}}\Delta\dot{r} - I_{\mathrm{xz}}\Delta\dot{p} &= (N_v \Delta v + N_p \Delta p + N_r \Delta r + N_{\delta a}\Delta\delta a + N_{\delta r}\Delta\delta r)\,.
\end{aligned}$$

Per our assumptions, the asymmetric state and control variables do not appear in the equations for $\Delta\dot{u}$, $\Delta\dot{w}$, and $\Delta\dot{q}$. Conversely, the symmetric state and control variables do not appear in the equations for $\Delta\dot{v}$, $\Delta\dot{p}$, and $\Delta\dot{r}$.

Notice that the forcing on the right-hand side of the equations for $m\Delta\dot{w}$ and $I_{\mathrm{y}}\Delta\dot{q}$ include an aerodynamic contribution due to $\Delta\dot{w}$. Thus, to obtain a simple expression for $\Delta\dot{w}$, one must bring the right-hand side term involving $\Delta\dot{w}$ across to the left. To obtain a decoupled equation for $\Delta\dot{q}$, one must substitute the resulting expression for $\Delta\dot{w}$ on the right-hand side. Also notice the inertial coupling between the roll and yaw dynamic equations. One must decouple these two equations for $\Delta\dot{p}$ and $\Delta\dot{r}$ in order to express the linearized dynamics in their simplest form.

Linearized Longitudinal Equations. We may partition the state and control vector into longitudinal (or symmetric) state and control vectors and lateral-directional (or asymmetric) state and control vectors. The longitudinal state and control vectors are:

$$\boldsymbol{x}_{\mathrm{L}} = \begin{pmatrix} \Delta x \\ \Delta z \\ \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{u}_{\mathrm{L}} = \begin{pmatrix} \Delta\delta e \\ \Delta\delta T \end{pmatrix}.$$

The linearized longitudinal equations take the form

$$\dot{\boldsymbol{x}}_{\mathrm{L}} = \boldsymbol{A}_{\mathrm{L}}\boldsymbol{x}_{\mathrm{L}} + \boldsymbol{B}_{\mathrm{L}}\boldsymbol{u}_{\mathrm{L}}$$

where

$$
\boldsymbol{A}_{\mathrm{L}} = \left(\begin{array}{cc:cccc}
0 & 0 & \cos\theta_0 & \sin\theta_0 & 0 & -u_0\sin\theta_0 \\
0 & 0 & -\sin\theta_0 & \cos\theta_0 & 0 & -u_0\cos\theta_0 \\
\hdashline
0 & 0 & \frac{1}{m}X_u & \frac{1}{m}X_w & 0 & -g\cos\theta_0 \\
0 & 0 & \frac{Z_u}{m-Z_{\dot{w}}} & \frac{Z_w}{m-Z_{\dot{w}}} & \frac{(Z_q+mu_0)}{m-Z_{\dot{w}}} & -\frac{mg\sin\theta_0}{m-Z_{\dot{w}}} \\
0 & 0 & \frac{1}{I_{\mathrm{y}}}\left(M_u+\frac{M_{\dot{w}}Z_u}{m-Z_{\dot{w}}}\right) & \frac{1}{I_{\mathrm{y}}}\left(M_w+\frac{M_{\dot{w}}Z_w}{m-Z_{\dot{w}}}\right) & \frac{1}{I_{\mathrm{y}}}\left(M_q+\frac{M_{\dot{w}}(Z_q+mu_0)}{m-Z_{\dot{w}}}\right) & \frac{1}{I_{\mathrm{y}}}\left(\frac{M_{\dot{w}}(-mg\sin\theta_0)}{m-Z_{\dot{w}}}\right) \\
0 & 0 & 0 & 0 & 1 & 0
\end{array}\right)
$$

and

$$
\boldsymbol{B}_{\mathrm{L}} = \left(\begin{array}{cc}
0 & 0 \\
0 & 0 \\
\hdashline
\frac{1}{m}X_{\delta e} & \frac{1}{m}X_{\delta T} \\
\frac{Z_{\delta e}}{m-Z_{\dot{w}}} & \frac{Z_{\delta T}}{m-Z_{\dot{w}}} \\
\frac{1}{I_{\mathrm{y}}}\left(M_{\delta e}+\frac{M_{\dot{w}}Z_{\delta e}}{m-Z_{\dot{w}}}\right) & \frac{1}{I_{\mathrm{y}}}\left(M_{\delta T}+\frac{M_{\dot{w}}Z_{\delta T}}{m-Z_{\dot{w}}}\right) \\
0 & 0
\end{array}\right)
$$

The matrices are partitioned to emphasize that the state variables Δu, Δw, Δq, and $\Delta\theta$ evolve independently of the state variables Δx and Δz. The x and z position play no role in the dynamic equations (aside from a second order dependence of air density on altitude). In studying aircraft stability and control, one typically ignores horizontal and vertical position.

The stick-fixed longitudinal response (i.e., the response with $\Delta\delta e = 0$ and $\Delta\delta T = 0$) can be understood as the superposition of two oscillatory "modes" of motion. One mode, referred to as the *phugoid mode*, corresponds to a severely underdamped oscillation with a fairly large damped natural period. The other mode, referred to as the *short period mode*, corresponds to a well-damped oscillation with a relatively small damped natural period. Stability of these modes of motion determines longitudinal dynamic stability.

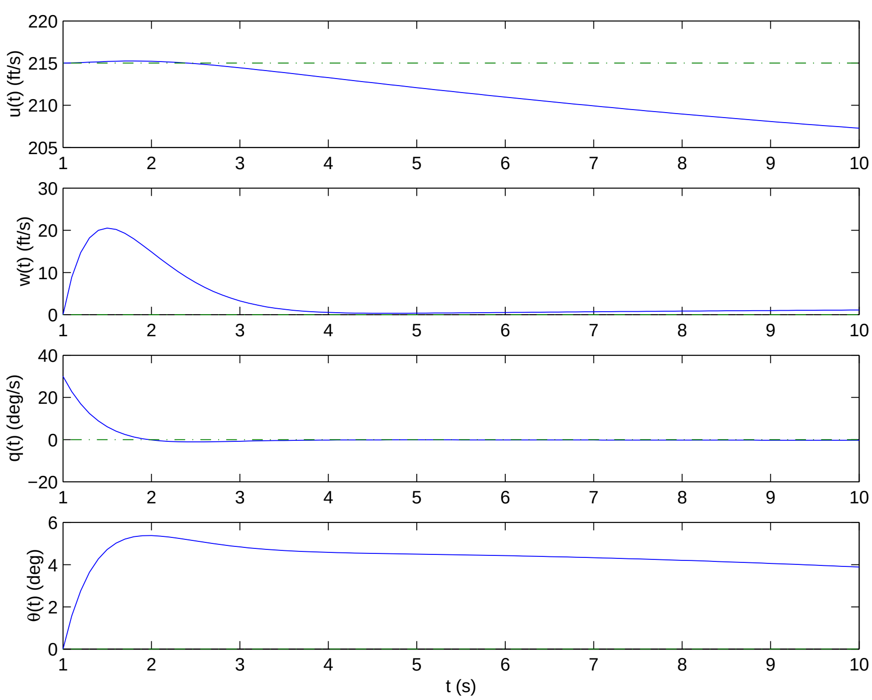

Figure 1: Short period contribution to the longitudinal response.

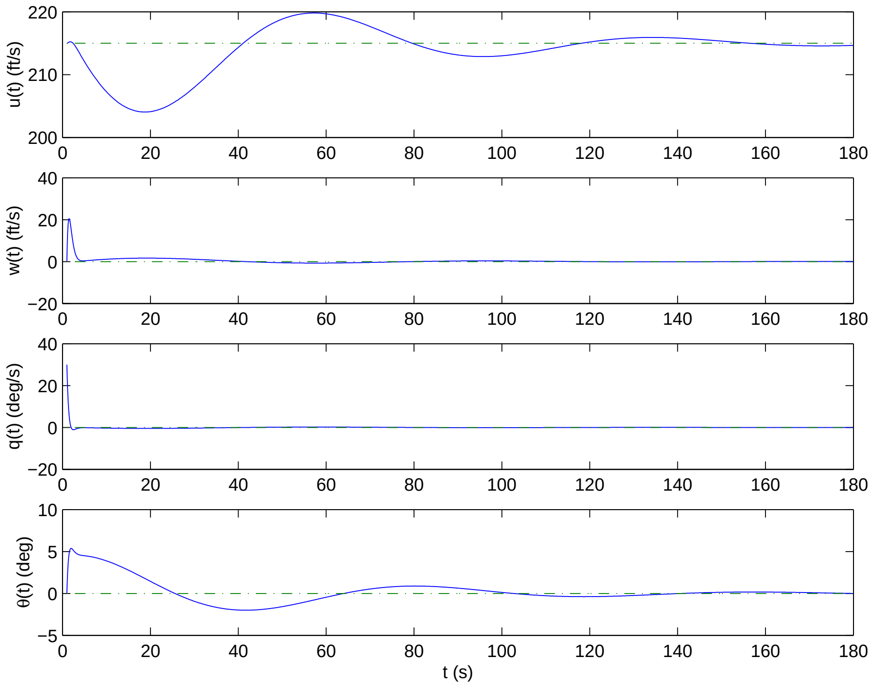

Figure 2: Long period (phugoid) contribution to the longitudinal response.

Figure 1 shows a longitudinal time history for a particular short takeoff and landing (STOL) airplane in response to an impulsive pitch disturbance (modeled as a non-zero initial pitch rate). Notice that w, which is proportional to the angle of attack for small perturbations, converges to near zero in only a few seconds. This quick convergence of the angle of attack is characteristic of a stable short period mode. Figure 2 shows a longer record of the time history for the same perturbation. Notice the much slower convergence of speed and pitch angle. This long period, lightly damped oscillation is characteristic of a stable long period, or phugoid, mode.

Aside: Approximating the Phugoid Mode. In W. F. Lanchester's original 1908 investigation of the phugoid mode, he considered the nonlinear dynamic equations for an airplane in longitudinal flight. He made three important assumptions:

- The angle of attack remains identically zero,
- thrust exactly balances drag throughout the motion, and
- the lift coefficient is constant and equal to nominal weight coefficient.

Let $\theta_0 = 0$. Moreover, assume that Z_q and $Z_{\dot{w}}$ are negligibly small. In the case of small perturbations, the assumption that $\Delta\alpha \equiv 0$ means that

$$\Delta w \equiv 0 \qquad \Rightarrow \qquad \Delta\dot{w} \equiv 0 = \frac{1}{m} Z_u \Delta u + u_0 \Delta q \qquad \Rightarrow \qquad \Delta q = -\frac{Z_u}{m u_0} \Delta u.$$

The linearized equation for $\Delta\dot{w}$ is trivial, under the given assumptions. The equation for $\Delta\dot{q}$ is simply a scaling of the equation for $\Delta\dot{u}$, under the given assumptions. Both equations may therefore be ignored. The remaining two first order equations which approximate the phugoid mode are therefore

$$\Delta\dot{u} = \frac{1}{m} X_u \Delta u - g\Delta\theta$$

$$\begin{aligned}\Delta\dot{\theta} &= \Delta q \\ &= -\frac{Z_u}{mu_0}\Delta u.\end{aligned}$$

Solving the latter equation for Δu and substituting into the former equation gives a single second order ODE:

$$\Delta\ddot{\theta} - \frac{X_u}{m}\Delta\dot{\theta} - \frac{gZ_u}{mu_0}\Delta\theta = 0.$$

This equation represents a damped linear oscillator (e.g., a mass-spring-damper system). Assuming that the discriminant

$$\left(-\frac{X_u}{m}\right)^2 - 4\left(-\frac{gZ_u}{mu_0}\right)$$

is negative, the system is underdamped and we may compute the approximate phugoid natural frequency and damping ratio:

$$\omega_{n_\mathrm{P}} \approx \sqrt{-\frac{Z_u g}{mu_0}} \qquad \text{and} \qquad \zeta_\mathrm{P} \approx -\frac{X_u}{2m\omega_{n_\mathrm{P}}}.$$

Although we will not discuss methods for computing dimensional stability derivatives, such as X_u and Z_u, until the next lecture, one may show that in the case of a jet, where thrust is independent of speed,

$$\omega_{n_\mathrm{P}} \approx \sqrt{2}\frac{g}{u_0} \qquad \text{and} \qquad \zeta_\mathrm{P} \approx \frac{1}{\sqrt{2}}\frac{1}{L/D}.$$

A higher nominal speed thus corresponds to a lower phugoid frequency (that is, a longer period of oscillation). A higher lift-to-drag ratio corresponds to lower damping. A derivation of this incompressible phugoid mode approximation, and a very nice discussion of Lanchester's original phugoid theory, are given in [1].

We will revisit this approximation for the phugoid mode, and develop another for the short period mode, after we have discussed methods for computing dimensional stability derivatives.

Linearized Lateral-Directional Equations. The lateral-directional state and control vectors are:

$$\boldsymbol{x}_\mathrm{LD} = \begin{pmatrix} \Delta y \\ \Delta\psi \\ \Delta v \\ \Delta p \\ \Delta r \\ \Delta\phi \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{u}_\mathrm{LD} = \begin{pmatrix} \Delta\delta a \\ \Delta\delta r \end{pmatrix}.$$

Let

$$\xi = I_\mathrm{x}I_\mathrm{z} - I_\mathrm{xz}^2.$$

The linearized lateral-directional equations take the form

$$\dot{\boldsymbol{x}}_\mathrm{LD} = \boldsymbol{A}_\mathrm{LD}\boldsymbol{x}_\mathrm{LD} + \boldsymbol{B}_\mathrm{LD}\boldsymbol{u}_\mathrm{LD}$$

where

$$\boldsymbol{A}_\mathrm{LD} = \left(\begin{array}{cc:cccc} 0 & u_0\cos\theta_0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sec\theta_0 & 0 \\ \hdashline 0 & 0 & \frac{1}{m}Y_v & \frac{1}{m}Y_p & \frac{1}{m}Y_r - u_0 & g\cos\theta_0 \\ 0 & 0 & \frac{1}{\xi}\left(I_\mathrm{z}L_v + I_\mathrm{xz}N_v\right) & \frac{1}{\xi}\left(I_\mathrm{z}L_p + I_\mathrm{xz}N_p\right) & \frac{1}{\xi}\left(I_\mathrm{z}L_r + I_\mathrm{xz}N_r\right) & 0 \\ 0 & 0 & \frac{1}{\xi}\left(I_\mathrm{xz}L_v + I_\mathrm{x}N_v\right) & \frac{1}{\xi}\left(I_\mathrm{xz}L_p + I_\mathrm{x}N_p\right) & \frac{1}{\xi}\left(I_\mathrm{xz}L_r + I_\mathrm{x}N_r\right) & 0 \\ 0 & 0 & 0 & 1 & \tan\theta_0 & 0 \end{array}\right)$$

and

$$
\boldsymbol{B}_{\mathrm{LD}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \hline \frac{1}{m} Y_{\delta a} & \frac{1}{m} Y_{\delta r} \\ \frac{1}{\xi}\left(I_{\mathrm{z}} L_{\delta a} + I_{\mathrm{xz}} N_{\delta a}\right) & \frac{1}{\xi}\left(I_{\mathrm{z}} L_{\delta r} + I_{\mathrm{xz}} N_{\delta r}\right) \\ \frac{1}{\xi}\left(I_{\mathrm{xz}} L_{\delta a} + I_{\mathrm{x}} N_{\delta a}\right) & \frac{1}{\xi}\left(I_{\mathrm{xz}} L_{\delta r} + I_{\mathrm{x}} N_{\delta r}\right) \\ 0 & 0 \end{pmatrix}.
$$

The matrices are partitioned to emphasize that the state variables Δv, Δp, Δr, and $\Delta\phi$ evolve independently of the state variable Δy and $\Delta\psi$. That is, y position and heading play no role in the dynamic equations. In studying aircraft stability and control, one typically ignores lateral position and heading.

If the stability axes happen to coincide with the principal axes of inertia, then $I_{\mathrm{xz}} = 0$. In this case, the linearized dynamic equations (less the equations for Δy and $\Delta\psi$) simplify to

$$
\frac{d}{dt}\begin{pmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta\phi \end{pmatrix} = \begin{pmatrix} \frac{1}{m} Y_v & \frac{1}{m} Y_p & \frac{1}{m} Y_r - u_0 & g\cos\theta_0 \\ \frac{1}{I_{\mathrm{x}}} L_v & \frac{1}{I_{\mathrm{x}}} L_p & \frac{1}{I_{\mathrm{x}}} L_r & 0 \\ \frac{1}{I_{\mathrm{z}}} N_v & \frac{1}{I_{\mathrm{z}}} N_p & \frac{1}{I_{\mathrm{z}}} N_r & 0 \\ 0 & 1 & \tan\theta_0 & 0 \end{pmatrix}\begin{pmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta\phi \end{pmatrix} + \begin{pmatrix} \frac{1}{m} Y_{\delta a} & \frac{1}{m} Y_{\delta r} \\ \frac{1}{I_{\mathrm{x}}} L_{\delta a} & \frac{1}{I_{\mathrm{x}}} L_{\delta r} \\ \frac{1}{I_{\mathrm{z}}} N_{\delta a} & \frac{1}{I_{\mathrm{z}}} N_{\delta r} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \Delta\delta a \\ \Delta\delta r \end{pmatrix}.
$$

Notice that, while the *inertial coupling* between roll rate and yaw rate vanishes in this case, the *aerodynamic coupling* does not; terms like L_r, N_p, $L_{\delta r}$ and $N_{\delta a}$ are not necessarily zero.

We will see shortly that the stick-fixed lateral-directional response (i.e., the response with $\Delta\delta a = 0$ and $\Delta\delta r = 0$) can be understood as the superposition of two non-oscillatory modes of motion (the *spiral* and *roll modes*) and an oscillatory mode (the *Dutch roll mode*). Stability of these modes of motion determines lateral-directional dynamic stability.

Recap. At this point, we have obtained twelve first order, linear time-invariant ordinary differential equations describing the motion of an airplane in response to small perturbations from wings-level equilibrium flight. Under certain assumptions, these twelve equations decouple into two sets of six ODE's. Moreover, in considering *dynamic stability*, we may ignore the variables Δx, Δy, Δz and $\Delta\psi$ and study only two sets of four ODE's: the (reduced) longitudinal equations and the (reduced) lateral-directional equations. First, though, we must understand how to relate estimates of aerodynamic properties such as lift, drag and pitching moment to dimensional stability derivatives such as X_u, $Z_{\dot{w}}$ and M_q.

References

[1] B. Etkin and L. D. Reid. *Dynamics of Flight: Stability and Control.* John Wiley and Sons, New York, NY, third edition, 1996.