Lecture 14: Small Disturbance Equations of Motion

We are considering a rigid airplane with a coordinate frame fixed at the center of gravity such that the xz-plane is a plane of symmetry. Written in these coordinates, the kinematic equations are

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \boldsymbol{R}_{\mathrm{IB}}\boldsymbol{v} = \begin{pmatrix} \cos\theta\cos\psi & \cos\psi\sin\theta\sin\phi - \cos\phi\sin\psi & \cos\psi\sin\theta\cos\phi + \sin\phi\sin\psi \\ \cos\theta\sin\psi & \cos\phi\cos\psi + \sin\theta\sin\phi\sin\psi & -\sin\phi\cos\psi + \sin\theta\cos\phi\sin\psi \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \boldsymbol{L}_{\mathrm{IB}}\boldsymbol{\omega} = \begin{pmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

and the dynamic equations are

$$m\dot{\boldsymbol{v}} = m\boldsymbol{v}\times\boldsymbol{\omega} + \begin{pmatrix} X(\boldsymbol{v},\boldsymbol{\omega},\boldsymbol{u}) \\ Y(\boldsymbol{v},\boldsymbol{\omega},\boldsymbol{u}) \\ Z(\boldsymbol{v},\boldsymbol{\omega},\boldsymbol{u}) \end{pmatrix} + mg\begin{pmatrix} -\sin\theta \\ \cos\theta\sin\phi \\ \cos\theta\cos\phi \end{pmatrix}$$

$$\boldsymbol{I}\dot{\boldsymbol{\omega}} = \boldsymbol{I}\boldsymbol{\omega}\times\boldsymbol{\omega} + \begin{pmatrix} L(\boldsymbol{v},\boldsymbol{\omega},\boldsymbol{u}) \\ M(\boldsymbol{v},\boldsymbol{\omega},\boldsymbol{u}) \\ N(\boldsymbol{v},\boldsymbol{\omega},\boldsymbol{u}) \end{pmatrix}.$$

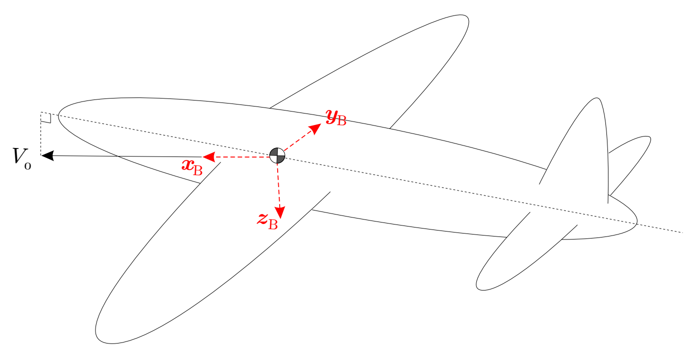

Figure 1: Stability Axes.

For the purpose of defining the "small disturbance" equations of motion, we will assume that a nominal wings-level, equilibrium flight condition has been identified and that the *stability axes* serve as the body-fixed reference frame. We will substitute the following values into the equations of motion

$$\begin{array}{lll} x = x_0(t) + \Delta x & y = y_0(t) + \Delta y & z = z_0(t) + \Delta z \\ \phi = \phi_0 + \Delta\phi & \theta = \theta_0 + \Delta\theta & \psi = \psi_0 + \Delta\psi \\ u = u_0 + \Delta u & v = v_0 + \Delta v & w = w_0 + \Delta w \\ p = p_0 + \Delta p & q = q_0 + \Delta q & r = r_0 + \Delta r \\ X = X_0 + \Delta X & Y = Y_0 + \Delta Y & Z = Z_0 + \Delta Z \\ L = L_0 + \Delta L & M = M_0 + \Delta M & N = N_0 + \Delta N. \end{array}$$

In stability axes, we have

$$\phi_0 \equiv 0, \quad \psi_0 \equiv 0, \quad v_0 \equiv 0, \quad w_0 \equiv 0, \quad p_0 \equiv 0, \quad q_0 \equiv 0, \quad r_0 \equiv 0.$$

The nominal values of the remaining variables (for example, u and θ) are generically nonzero. (The assumption $\psi_0 = 0$ is entirely arbitrary because ψ does not appear anywhere in the equations of motion. This choice corresponds, for example, to "due north" flight.)

Linearized Kinematics. We will assume small perturbations so that the linearized equations accurately approximate the nonlinear ones. To start, take the first view of linearization, as discussed in the previous lecture, and consider only the kinematic equation for position x. Define f_1 such that

$$\begin{aligned}\dot{x} &= f_1(x, y, z, \phi, \theta, \psi, u, v, w, p, q, r, \delta T, \delta a, \delta e, \delta r) \\ &= (\cos\theta\cos\psi)\, u + (\cos\psi\sin\theta\sin\phi - \cos\phi\sin\psi)\, v + (\cos\psi\sin\theta\cos\phi + \sin\phi\sin\psi)\, w \end{aligned} \tag{1}$$

Note that f_1 is the first component of the twelve-dimensional vector field $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$ that defines the airplane equations of motion. To linearize, compute

$$\begin{aligned}\Delta\dot{x} &= \frac{\partial f_1}{\partial \boldsymbol{x}}\boldsymbol{\Delta x} \\ &= \left.\frac{\partial f_1}{\partial \phi}\right|_0 \Delta\phi + \left.\frac{\partial f_1}{\partial \theta}\right|_0 \Delta\theta + \left.\frac{\partial f_1}{\partial \psi}\right|_0 \Delta\psi + \left.\frac{\partial f_1}{\partial u}\right|_0 \Delta u + \left.\frac{\partial f_1}{\partial v}\right|_0 \Delta v + \left.\frac{\partial f_1}{\partial w}\right|_0 \Delta w \\ &= -u_0\sin\theta_0\Delta\theta + \cos\theta_0\Delta u + \sin\theta_0\Delta w.\end{aligned}$$

Alternatively, using the second view of linearization, we may use small angle approximations and neglect products of perturbation variables. We may write

$$\begin{aligned}\cos\phi = \cos(\Delta\phi) &\approx 1 \\ \sin\phi = \sin(\Delta\phi) &\approx \Delta\phi \\ \cos\theta = \cos(\theta_0 + \Delta\theta) &= \cos\theta_0\cos\Delta\theta - \sin\theta_0\sin\Delta\theta \\ &\approx \cos\theta_0 - \sin\theta_0\Delta\theta \\ \sin\theta = \sin(\theta_0 + \Delta\theta) &= \sin\theta_0\cos\Delta\theta + \cos\theta_0\sin\Delta\theta \\ &\approx \sin\theta_0 + \cos\theta_0\Delta\theta \\ \cos\psi = \cos(\Delta\psi) &\approx 1 \\ \sin\psi = \sin(\Delta\psi) &\approx \Delta\phi.\end{aligned}$$

Substituting the above approximations, along with the identities $u = u_0 + \Delta u$, $v = \Delta v$, and $w = \Delta w$, into (1) and neglecting perturbation terms higher than first order gives

$$\dot{x}_0 + \Delta\dot{x} \approx (u_0 + \Delta u)\cos\theta_0 - u_0\sin\theta_0\Delta\theta + \sin\theta_0\Delta w.$$

Following this approach for each of the kinematic equations, we find that the first-order approximation to the complete set of kinematic equations is

$$\begin{aligned}\dot{x}_0 + \Delta\dot{x} &= (u_0 + \Delta u)\cos\theta_0 - (u_0\sin\theta_0)\Delta\theta + \sin\theta_0\Delta w \\ \dot{y}_0 + \Delta\dot{y} &= (u_0\cos\theta_0)\Delta\psi + \Delta v \\ \dot{z}_0 + \Delta\dot{z} &= -(u_0 + \Delta u)\sin\theta_0 - (u_0\cos\theta_0)\Delta\theta + \cos\theta_0\Delta w \\ \dot{\phi}_0 + \Delta\dot{\phi} &= \Delta p + \tan\theta_0\Delta r \\ \dot{\theta}_0 + \Delta\dot{\theta} &= \Delta q \\ \dot{\psi}_0 + \Delta\dot{\psi} &= \sec\theta_0\Delta r.\end{aligned}$$

Setting the perturbation values to zero leaves the nominal equations. Subtracting these from the complete equations then gives the perturbed equations. The resulting nominal and perturbation equations for the vehicle kinematics are:

$$
\begin{aligned}
\dot{x}_0 &= u_0 \cos\theta_0 \\
\dot{y}_0 &= 0 \\
\dot{z}_0 &= -u_0 \sin\theta_0 \\
\dot{\phi}_0 &= 0 \\
\dot{\theta}_0 &= 0 \\
\dot{\psi}_0 &= 0
\end{aligned}
\qquad \text{and} \qquad
\begin{aligned}
\Delta\dot{x} &= \cos\theta_0 \Delta u - (u_0 \sin\theta_0)\Delta\theta + \sin\theta_0 \Delta w \\
\Delta\dot{y} &= (u_0 \cos\theta_0)\Delta\psi + \Delta v \\
\Delta\dot{z} &= -\sin\theta_0 \Delta u - (u_0 \cos\theta_0)\Delta\theta + \cos\theta_0 \Delta w \\
\Delta\dot{\phi} &= \Delta p + \tan\theta_0 \Delta r \\
\Delta\dot{\theta} &= \Delta q \\
\Delta\dot{\psi} &= \sec\theta_0 \Delta r
\end{aligned}
$$

Linearized Dynamics. First, consider the translational dynamics. Written explicitly, we have

$$
m \begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = m \begin{pmatrix} u \\ v \\ w \end{pmatrix} \times \begin{pmatrix} p \\ q \\ r \end{pmatrix} + \begin{pmatrix} X(\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{u}) \\ Y(\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{u}) \\ Z(\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{u}) \end{pmatrix} + mg \begin{pmatrix} -\sin\theta \\ \cos\theta \sin\phi \\ \cos\theta \cos\phi \end{pmatrix}.
$$

Substituting nominal-plus-perturbed values for each component of the system state gives

$$
m \begin{pmatrix} \Delta\dot{u} \\ \Delta\dot{v} \\ \Delta\dot{w} \end{pmatrix} = m \left[\begin{pmatrix} u_0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \Delta u \\ \Delta v \\ \Delta w \end{pmatrix} \right] \times \begin{pmatrix} \Delta p \\ \Delta q \\ \Delta r \end{pmatrix} + \begin{pmatrix} X_0 + \Delta X \\ Y_0 + \Delta Y \\ Z_0 + \Delta Z \end{pmatrix} + mg \begin{pmatrix} -\sin(\theta_0 + \Delta\theta) \\ \cos(\theta_0 + \Delta\theta) \sin\Delta\phi \\ \cos(\theta_0 + \Delta\theta) \cos\Delta\phi \end{pmatrix}.
$$

Ignoring higher order perturbation terms leaves the following first order approximate equations:

$$
\begin{aligned}
m\Delta\dot{u} &= X_0 + \Delta X - mg(\sin\theta_0 + \cos\theta_0 \Delta\theta) \\
m\Delta\dot{v} &= -mu_0 \Delta r + Y_0 + \Delta Y + mg \cos\theta_0 \Delta\phi \\
m\Delta\dot{w} &= mu_0 \Delta q + Z_0 + \Delta Z + mg(\cos\theta_0 - \sin\theta_0 \Delta\theta)
\end{aligned}
$$

Turning next to the rotational dynamics, we have

$$
\begin{pmatrix} I_x & 0 & -I_{xz} \\ 0 & I_y & 0 \\ -I_{xz} & 0 & I_z \end{pmatrix} \begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \left[\begin{pmatrix} I_x & 0 & -I_{xz} \\ 0 & I_y & 0 \\ -I_{xz} & 0 & I_z \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix} \right] \times \begin{pmatrix} p \\ q \\ r \end{pmatrix} + \begin{pmatrix} L(\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{u}) \\ M(\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{u}) \\ N(\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{u}) \end{pmatrix}.
$$

Substituting nominal-plus-perturbed values for each component of the system state gives

$$
\begin{pmatrix} I_x & 0 & -I_{xz} \\ 0 & I_y & 0 \\ -I_{xz} & 0 & I_z \end{pmatrix} \begin{pmatrix} \Delta\dot{p} \\ \Delta\dot{q} \\ \Delta\dot{r} \end{pmatrix} = \left[\begin{pmatrix} I_x & 0 & -I_{xz} \\ 0 & I_y & 0 \\ -I_{xz} & 0 & I_z \end{pmatrix} \begin{pmatrix} \Delta p \\ \Delta q \\ \Delta r \end{pmatrix} \right] \times \begin{pmatrix} \Delta p \\ \Delta q \\ \Delta r \end{pmatrix} + \begin{pmatrix} L_0 + \Delta L \\ M_0 + \Delta M \\ N_0 + \Delta N \end{pmatrix}.
$$

Ignoring higher order perturbation terms leaves the following first order approximate equations:

$$
\begin{aligned}
I_{\text{x}}\Delta\dot{p} - I_{\text{xz}}\Delta\dot{r} &= L_0 + \Delta L \\
I_{\text{y}}\Delta\dot{q} &= M_0 + \Delta M \\
I_{\text{z}}\Delta\dot{r} - I_{\text{xz}}\Delta\dot{p} &= N_0 + \Delta N.
\end{aligned}
$$

Setting the perturbation values to zero leaves the nominal equations. Subtracting these from the complete equations then gives the perturbed equations. The resulting nominal and perturbation equations for the vehicle dynamics are:

$$
\begin{aligned}
0 &= X_0 - mg \sin\theta_0 \\
0 &= Y_0 \\
0 &= Z_0 + mg \cos\theta_0 \\
0 &= L_0 \\
0 &= M_0 \\
0 &= N_0.
\end{aligned}
\qquad \text{and} \qquad
\begin{aligned}
m\Delta\dot{u} &= \Delta X - mg \cos\theta_0 \Delta\theta \\
m\Delta\dot{v} &= -mu_0 \Delta r + \Delta Y + mg \cos\theta_0 \Delta\phi \\
m\Delta\dot{w} &= mu_0 \Delta q + \Delta Z - mg \sin\theta_0 \Delta\theta \\
I_{\text{x}}\Delta\dot{p} - I_{\text{xz}}\Delta\dot{r} &= \Delta L \\
I_{\text{y}}\Delta\dot{q} &= \Delta M \\
I_{\text{z}}\Delta\dot{r} - I_{\text{xz}}\Delta\dot{p} &= \Delta N.
\end{aligned}
$$

The challenge at this point is to express the aerodynamic forces and moments which result from perturbations from the nominal flight condition. The aerodynamic forces and moments developed over a moving rigid body are functions of the body's geometry, the local density of air, the control surface deflections, the air-relative velocity, and, in general, the entire time history of the body's motion. In practice, however, the following assumptions are well-justified for most flight conditions:

- The asymmetric force Y and moments L, and N are, to first order, independent of the symmetric state and control variables. That is, the derivatives of Y, L, and N with respect to θ, u, w, q, δT, and δe are identically zero.
- The symmetric forces X and Z and moment M are, to first order, independent of the asymmetric state and control variables. That is, the derivatives of X, Z, and M with respect to ϕ, ψ, v, p, r, δa and δr are identically zero.
- The only dependence of the aerodynamic forces and moments on acceleration is the dependence of Z and M on $\dot{w}$. (Because $\Delta\dot{w} \approx V_0\Delta\dot{\alpha} = u_0\Delta\dot{\alpha}$, these terms essentially account for the dependence of lift and moment on the rate of change of angle of attack.)
- The force X is independent of pitch rate q.
- Density remains constant over the range of perturbations.

Define the dimensional derivatives

$$X_{(\cdot)} = \left.\frac{\partial X}{\partial(\cdot)}\right|_0, \qquad Y_{(\cdot)} = \left.\frac{\partial Y}{\partial(\cdot)}\right|_0, \qquad \text{and} \qquad Z_{(\cdot)} = \left.\frac{\partial Z}{\partial(\cdot)}\right|_0.$$

(For example, let $X_u = \frac{\partial X}{\partial u}$.) Also, define

$$L_{(\cdot)} = \left.\frac{\partial L}{\partial(\cdot)}\right|_0, \qquad M_{(\cdot)} = \left.\frac{\partial M}{\partial(\cdot)}\right|_0, \qquad \text{and} \qquad N_{(\cdot)} = \left.\frac{\partial N}{\partial(\cdot)}\right|_0.$$

With the assumptions and definitions above, we may rewrite the linearized dynamic equations as

$$\begin{aligned}
m\Delta\dot{u} &= (X_u\Delta u + X_w\Delta w + X_{\delta e}\Delta\delta e + X_{\delta T}\Delta\delta T) - mg\cos\theta_0\Delta\theta \\
m\Delta\dot{v} &= -mu_0\Delta r + (Y_v\Delta v + Y_p\Delta p + Y_r\Delta r + Y_{\delta a}\Delta\delta a + Y_{\delta r}\Delta\delta r) + mg\cos\theta_0\Delta\phi \\
m\Delta\dot{w} &= mu_0\Delta q + (Z_u\Delta u + Z_w\Delta w + Z_{\dot{w}}\Delta\dot{w} + Z_q\Delta q + Z_{\delta e}\Delta\delta e + Z_{\delta T}\Delta\delta T) - mg\sin\theta_0\Delta\theta \\
I_{\text{x}}\Delta\dot{p} - I_{\text{xz}}\Delta\dot{r} &= (L_v\Delta v + L_p\Delta p + L_r\Delta r + L_{\delta a}\Delta\delta a + L_{\delta r}\Delta\delta r) \\
I_{\text{y}}\Delta\dot{q} &= (M_u\Delta u + M_w\Delta w + M_{\dot{w}}\Delta\dot{w} + M_q\Delta q + M_{\delta e}\Delta\delta e + M_{\delta T}\Delta\delta T) \\
I_{\text{z}}\Delta\dot{r} - I_{\text{xz}}\Delta\dot{p} &= (N_v\Delta v + N_p\Delta p + N_r\Delta r + N_{\delta a}\Delta\delta a + N_{\delta r}\Delta\delta r)\,.
\end{aligned}$$

Per our assumptions, the asymmetric state and control variables do not appear in the equations for $\Delta\dot{u}$, $\Delta\dot{w}$, and $\Delta\dot{q}$. Conversely, the symmetric state and control variables do not appear in the equations for $\Delta\dot{v}$, $\Delta\dot{p}$, and $\Delta\dot{r}$.