Lecture 13: Linearization

Consider a set of nonlinear, time-invariant ODE's. Recall that any n^{th} order ODE can be re-written as n first order ODE's. We may therefore assume, without loss of generality, that the system takes the form

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}), \tag{1}$$

where $\boldsymbol{f}$ is a vector-valued function of the *state* $\boldsymbol{x}$ and *input* $\boldsymbol{u}$. While we may well know the form of the input $\boldsymbol{u}$, we do not necessarily know the resulting form of $\boldsymbol{x}(t)$, for a given initial state $\boldsymbol{x}(0)$. A *solution* or *trajectory* of (1) is a pair $(\tilde{\boldsymbol{x}}(t), \tilde{\boldsymbol{u}}(t))$ satisfying

$$\dot{\tilde{\boldsymbol{x}}} = \boldsymbol{f}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{u}}).$$

Some solutions are special in the sense that they do not depend on time. An *equilibrium* is a solution $(\tilde{\boldsymbol{x}}(t), \tilde{\boldsymbol{u}}(t)) = (\boldsymbol{x}_{\mathrm{e}}, \boldsymbol{u}_{\mathrm{e}})$ for which $\boldsymbol{x}_{\mathrm{e}}$ and $\boldsymbol{u}_{\mathrm{e}}$ are *constant*, that is

$$\dot{\boldsymbol{x}}_{\mathrm{e}} = \boldsymbol{0} = \boldsymbol{f}(\boldsymbol{x}_{\mathrm{e}}, \boldsymbol{u}_{\mathrm{e}}).$$

To determine equilibria for a given system, one sets $\boldsymbol{f}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{u}})$ equal to zero and solves the resulting nonlinear algebraic equations for $\boldsymbol{x} = \boldsymbol{x}_{\mathrm{e}}$ and $\boldsymbol{u} = \boldsymbol{u}_{\mathrm{e}}$.

Linearization: View #1. Given a solution, and assuming that the components of $\boldsymbol{f}$ are suitably smooth, one may expand the right-hand side of equation (1) in a *multivariable Taylor series.* Doing so, one obtains

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{u}}) + \left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right)^{*} (\boldsymbol{x} - \tilde{\boldsymbol{x}}) + \left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\right)^{*} (\boldsymbol{u} - \tilde{\boldsymbol{u}}) + \text{h.o.t.}, \tag{2}$$

where the asterisk denotes that the quantity is evaluated along the solution $(\tilde{\boldsymbol{x}}(t), \tilde{\boldsymbol{u}}(t))$. So far, we have said nothing new. Equation (2) is an equivalent representation of equation (1). Rearranging (2), we obtain

$$\dot{\boldsymbol{x}} - \dot{\tilde{\boldsymbol{x}}} = \left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right)^{*} (\boldsymbol{x} - \tilde{\boldsymbol{x}}) + \left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\right)^{*} (\boldsymbol{u} - \tilde{\boldsymbol{u}}) + \text{h.o.t.},$$

where we have used the fact that $(\tilde{\boldsymbol{x}}(t), \tilde{\boldsymbol{u}}(t))$ is a solution in order to replace $\boldsymbol{f}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{u}})$ with $\dot{\tilde{\boldsymbol{x}}}$. The "linearization" occurs when we assume that $\|(\boldsymbol{x} - \tilde{\boldsymbol{x}})\|$ and $\|(\boldsymbol{u} - \tilde{\boldsymbol{u}})\|$ remain *small*, so that we may neglect the higher order terms in the Taylor series expansion. We make the following definitions:

$$\begin{aligned}
\boldsymbol{\Delta x} &= \boldsymbol{x} - \tilde{\boldsymbol{x}} \\
\boldsymbol{\Delta u} &= \boldsymbol{u} - \tilde{\boldsymbol{u}} \\
\boldsymbol{A}(t) &= \left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right)^{*} \\
\boldsymbol{B}(t) &= \left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\right)^{*}
\end{aligned}$$

The linearized state equation is

$$\frac{d}{dt}\boldsymbol{\Delta x} = \boldsymbol{A}(t)\boldsymbol{\Delta x} + \boldsymbol{B}(t)\boldsymbol{\Delta u}$$

So long as $\boldsymbol{\Delta x}$ and $\boldsymbol{\Delta u}$ remain "small enough," these equations will provide a good approximation to equations (1). Note that the state itself need not remain small. Only the difference between its actual and "nominal" values must remain small.

In general, the *state matrix* $\boldsymbol{A}$ and the *input matrix* $\boldsymbol{B}$ will depend on time explicitly. Even though the original nonlinear system is time-invariant, the dependence of the solution $(\tilde{\boldsymbol{x}}(t), \tilde{\boldsymbol{u}}(t))$ on time will carry

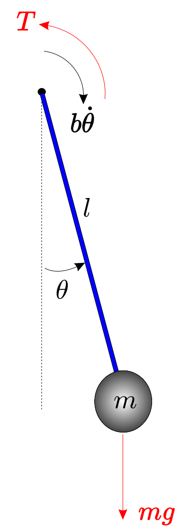

Figure 1: A planar pendulum with linear damping and an input torque.

through to the components of $\boldsymbol{A}$ and $\boldsymbol{B}$. Linearizing about an *equilibrium* $(\boldsymbol{x}_\mathrm{e}, \boldsymbol{u}_\mathrm{e})$, however, gives n first order, linear *time-invariant* ODE's

$$\frac{d}{dt}\boldsymbol{\Delta x} = \boldsymbol{A}\boldsymbol{\Delta x} + \boldsymbol{B}\boldsymbol{\Delta u}.$$

Example. Consider the planar pendulum shown in Figure 1. Summing moments about the pivot gives the following equation of motion

$$ml^2\ddot{\theta} = -mgl\sin\theta - b\dot{\theta} + T.$$

Letting

$$\boldsymbol{x} = \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix} \qquad \text{and} \qquad u = T,$$

we may rewrite this second order, nonlinear, time-invariant ODE as two first order equations

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -\frac{g}{l}\sin x_1 - \frac{b}{ml^2}x_2 + \frac{1}{ml^2}u \end{pmatrix}. \tag{3}$$

More compactly, we say

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, u)$$

where

$$\boldsymbol{f}(\boldsymbol{x}, u) = \begin{pmatrix} x_2 \\ -\frac{g}{l}\sin x_1 - \frac{b}{ml^2}x_2 + \frac{1}{ml^2}u \end{pmatrix}.$$

One equilibrium of the system (3) is

$$(\boldsymbol{x}_\mathrm{e}, u_\mathrm{e}) = (\boldsymbol{0}, 0). \tag{4}$$

The equilibrium (4) corresponds to the case where the pendulum hangs vertically downward at rest. (There is one other equilibrium for which $u_\mathrm{e} = 0$, corresponding to the pendulum standing vertically "on end.") Linearizing equations (3) about the equilibrium (4), as described above, we first obtain

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l}\cos x_1 & -\frac{b}{ml^2} \end{pmatrix} \qquad \text{and} \qquad \frac{\partial \boldsymbol{f}}{\partial u} = \begin{pmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{ml^2} \end{pmatrix}.$$

Evaluating these matrices at the equilibrium (4) gives

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{b}{ml^2} \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{B} = \begin{pmatrix} 0 \\ \frac{1}{ml^2} \end{pmatrix}.$$

Thus, the dynamics linearized about the equilibrium (4) are

$$\frac{d}{dt}\boldsymbol{\Delta x} = \boldsymbol{A}\boldsymbol{\Delta x} + \boldsymbol{B}\Delta u.$$

Written out explicitly, these equations are

$$\frac{d}{dt}\left(\begin{array}{c} \Delta x_1 \\ \Delta x_2 \end{array}\right) = \left(\begin{array}{c} \Delta x_2 \\ -\frac{g}{l}\Delta x_1 - \frac{b}{ml^2}\Delta x_2 + \frac{1}{ml^2}\Delta u \end{array}\right).$$

These are precisely the equations one would obtain by using the "small angle" approximation $\sin\theta \approx \theta$. Indeed, that approximation is simply the linearization of $\sin\theta$ about the angle $\theta = 0$.

Although the nonlinear equations (3) are difficult to solve analytically, one may use Matlab to compute a numerical solution for the pendulum response. Suppose the physical parameters are

$$m = 0.1 \text{ kg}, \qquad g = 10 \text{ m/s}^2, \qquad l = 1 \text{ m}, \qquad \text{and} \qquad b = 0 \text{ Nm/(rad/s)}.$$

Assume that no control torque is applied ($T = 0$), so that the response is due entirely to the initial condition, say:

$$\theta(0) = \theta_0 \qquad\qquad \dot{\theta}(0) = 0.$$

Consider the following three cases.

1. $\theta_0 = 20°$: Both systems oscillate at around $\sqrt{g/l}$ rad/s.
2. $\theta_0 = 179.9°$: The linear system oscillates at $\sqrt{g/l}$ rad/s but the nonlinear system takes a full 10 seconds to swing to the other side and back again.
3. $\theta_0 = 180°$: The linear system oscillates at $\sqrt{g/l}$ rad/s but the nonlinear system remains at the *inverted equilibrium*! Multiple isolated equilibria are a phenomenon exhibited only in nonlinear systems.

Except in the last example, damping would gradually bring θ to zero. However, the *transient* response for the two systems would remain drastically different for large θ. This example should underscore the fact that linearization only provides a valid approximation when the nonlinear system's motion stays "close" to the nominal motion. For the pendulum, this means that θ should remain small.

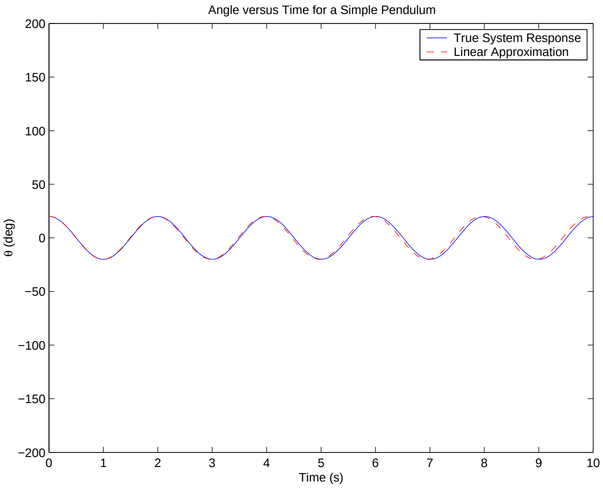

Figure 2: Initial Condition Response: $\theta_0 = 20°$.

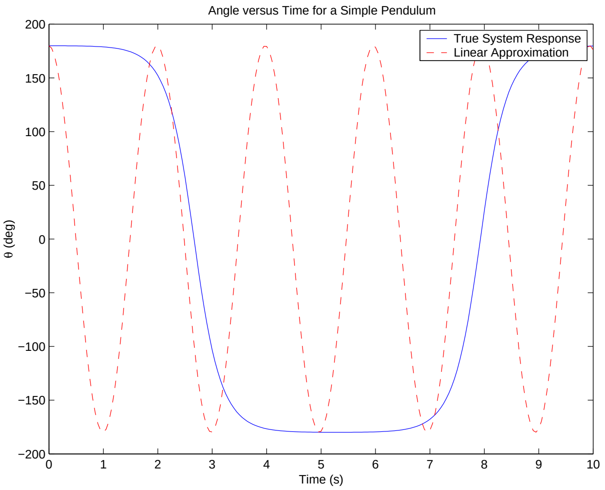

Figure 3: Initial Condition Response: $\theta_0 = 179.9°$.

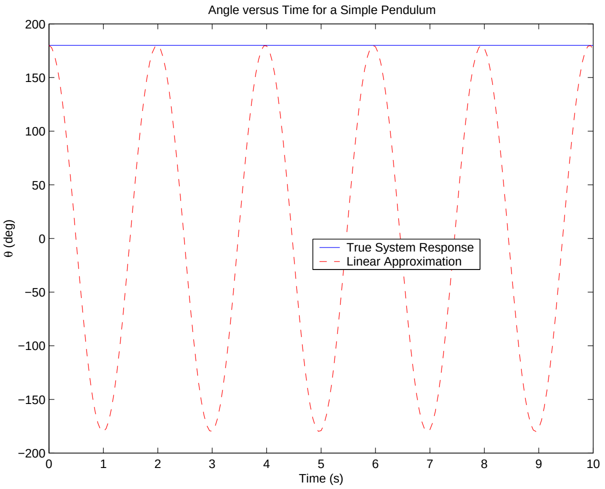

Figure 4: Initial Condition Response: $\theta_0 = 180°$.

Linearization: View #2. Another way to think about linearization is to suppose that a system is slightly disturbed from a nominal state of motion. In this view, the actual state $\boldsymbol{x}$ is the sum of the nominal state $\tilde{\boldsymbol{x}}$ and the state disturbance $\boldsymbol{\Delta x}$. Similarly, the actual input $\boldsymbol{u}$ is the sum of the nominal input $\tilde{\boldsymbol{u}}$ and the input disturbance $\boldsymbol{\Delta u}$.

For the pendulum equations

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -\frac{g}{l}\sin x_1 - \frac{b}{ml^2}x_2 + \frac{1}{ml^2}u \end{pmatrix},$$

let $\boldsymbol{x} = \tilde{\boldsymbol{x}} + \boldsymbol{\Delta x}$ and $u = \tilde{u} + \Delta u$, where $\tilde{\boldsymbol{x}}$ is a nominal value of the state (e.g., an equilibrium value) and $\tilde{u}$ is a nominal value of the input. Substituting above gives

$$\begin{aligned} \begin{pmatrix} \dot{\tilde{x}}_1 + \frac{d}{dt}\Delta x_1 \\ \dot{\tilde{x}}_2 + \frac{d}{dt}\Delta x_2 \end{pmatrix} &= \begin{pmatrix} \tilde{x}_2 + \Delta x_2 \\ -\frac{g}{l}\sin(\tilde{x}_1 + \Delta x_1) - \frac{b}{ml^2}(\tilde{x}_2 + \Delta x_2) + \frac{1}{ml^2}(\tilde{u} + \Delta u) \end{pmatrix} \\ &= \begin{pmatrix} \tilde{x}_2 + \Delta x_2 \\ -\frac{g}{l}\left(\cos\tilde{x}_1 \sin\Delta x_1 + \sin\tilde{x}_1\cos\Delta x_1\right) - \frac{b}{ml^2}(\tilde{x}_2 + \Delta x_2) + \frac{1}{ml^2}(\tilde{u} + \Delta u) \end{pmatrix}. \end{aligned}$$

The equations above are *equivalent* to the original equations. The "linearization" takes place when we assume that the disturbance is small. In that case, $\cos \Delta x_1 \approx 1$ and $\sin \Delta x_1 \approx \Delta x_1$ and the equations above can be re-written as

$$\begin{pmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{pmatrix} + \frac{d}{dt}\begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix} = \begin{pmatrix} \tilde{x}_2 \\ -\frac{g}{l}\sin \tilde{x}_1 - \frac{b}{ml^2}\tilde{x}_2 + \frac{1}{ml^2}\tilde{u} \end{pmatrix} + \begin{pmatrix} \Delta x_2 \\ -\frac{g}{l}\cos \tilde{x}_1 \Delta x_1 - \frac{b}{ml^2}\Delta x_2 + \frac{1}{ml^2}\Delta u \end{pmatrix}.$$

Now, the nominal state $\tilde{\boldsymbol{x}}$ and input $\tilde{u}$ about which we have linearized are a assumed to be a solution of the dynamic equations:

$$\begin{pmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{pmatrix} = \begin{pmatrix} \tilde{x}_2 \\ -\frac{g}{l}\sin \tilde{x}_1 - \frac{b}{ml^2}\tilde{x}_2 + \frac{1}{ml^2}\tilde{u} \end{pmatrix}.$$

Subtracting this equation from the one above leaves the "small disturbance" or "perturbation" equation

$$\frac{d}{dt}\begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix} = \begin{pmatrix} \Delta x_2 \\ -\frac{g}{l}\cos \tilde{x}_1 \Delta x_1 - \frac{b}{ml^2}\Delta x_2 + \frac{1}{ml^2}\Delta u \end{pmatrix}.$$

If we consider the nominal state $(\tilde{\boldsymbol{x}}, \tilde{u}) = (\mathbf{0}, 0)$, as before, then these equations are identical to the linearized equations obtained previously.

Nominal flight condition for a rigid airplane. Consider a rigid airplane with a reference frame fixed in the body at the center of gravity such that the xz-plane is a plane of symmetry. Written in this frame, the kinematic equations are

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} &= \boldsymbol{R}_{\text{IB}}(\phi, \theta, \psi)\boldsymbol{v} \\ \begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} &= \boldsymbol{L}_{\text{IB}}(\phi, \theta)\boldsymbol{\omega} \end{aligned}$$

and the dynamic equations are

$$\begin{aligned} m\dot{\boldsymbol{v}} + \boldsymbol{\omega} \times m\boldsymbol{v} &= \begin{pmatrix} X(\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{u}) \\ Y(\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{u}) \\ Z(\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{u}) \end{pmatrix} + mg\begin{pmatrix} -\sin\theta \\ \cos\theta\sin\phi \\ \cos\theta\cos\phi \end{pmatrix} \\ \boldsymbol{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \boldsymbol{I}\boldsymbol{\omega} &= \begin{pmatrix} L(\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{u}) \\ M(\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{u}) \\ N(\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{u}) \end{pmatrix}. \end{aligned}$$

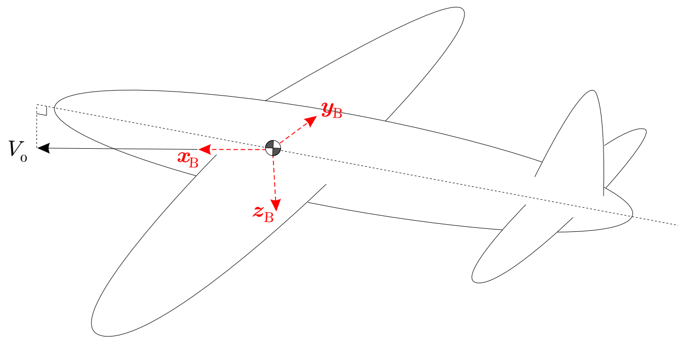

Figure 5: Stability Axes.

One of the fundamental questions to be answered, before one can linearize the aircraft equations of motion and analyze dynamic stability, is: "How should the body-fixed reference frame be oriented?" To start with, one should choose a body frame for the xz-plane as a plane of symmetry. However, this choice only fixes the body $\boldsymbol{y}_{\mathrm{B}}$ axis; the $\boldsymbol{x}_{\mathrm{B}}$-axis can point anywhere in the body longitudinal plane. For example, one could fix the frame such that the $\boldsymbol{x}_{\mathrm{B}}$-axis points out the nose of the aircraft, or along the aircraft zero-lift line. One could also choose the body frame to correspond to the *principal axes* of inertia, so that the inertia matrix $\boldsymbol{I}_{\mathrm{B}}$ is diagonal.

In studying dynamic stability, however, a more clever choice of body-fixed reference frame can significantly simplify the analysis. The *stability reference frame* (also known as *stability axes*) is defined relative to the wings-level equilibrium flight condition whose stability properties are being investigated. Specifically, *the stability frame is defined such that the body $\boldsymbol{x}_{\mathrm{B}}$-axis is aligned with the velocity vector when the aircraft is in its nominal flight condition.* In the stability reference frame, both v and w are zero in nominal flight. Note, however, that the reference frame is fixed in the body; the $\boldsymbol{x}_{\mathrm{B}}$-axis is *not* necessarily aligned with the velocity vector in perturbed flight.

For the purpose of defining the "small disturbance" equations of motion, we will assume that a nominal wings-level, equilibrium flight condition has been identified and that the stability axes serve as the body-fixed reference frame. We will substitute the following values into the equations of motion

$$\begin{array}{rclrclrcl}
x &=& x_0(t)+\Delta x & y &=& y_0(t)+\Delta y & z &=& z_0(t)+\Delta z \\
\phi &=& \phi_0+\Delta\phi & \theta &=& \theta_0+\Delta\theta & \psi &=& \psi_0+\Delta\psi \\
u &=& u_0+\Delta u & v &=& v_0+\Delta v & w &=& w_0+\Delta w \\
p &=& p_0+\Delta p & q &=& q_0+\Delta q & r &=& r_0+\Delta r \\
X &=& X_0+\Delta X & Y &=& Y_0+\Delta Y & Z &=& Z_0+\Delta Z \\
L &=& L_0+\Delta L & M &=& M_0+\Delta M & N &=& N_0+\Delta N.
\end{array}$$

Note that, because we are considering an equilibrium *motion*, the nominal position (given by x, y, and z) will change with time; we denote this dependence explicitly above. In stability axes, we have

$$y_0(t)\equiv 0,\quad \phi_0\equiv 0,\quad \psi_0\equiv 0,\quad v_0\equiv 0,\quad w_0\equiv 0,\quad p_0\equiv 0,\quad q_0\equiv 0,\quad r_0\equiv 0.$$

The nominal values of the remaining variables (for example, u and θ) are generically nonzero. (Note that the assumption $\psi_0=0$ is entirely arbitrary because ψ does not appear anywhere in the equations of motion. This choice corresponds, for example, to "due north" flight.) In the next lecture, we will proceed with the linearization procedure to obtain a set of first order, linear time-invariant equations which describe the motion of a rigid airplane operating "near" wings-level equilibrium flight.