

# Lecture 12: Rigid Body Dynamics & Kinematics

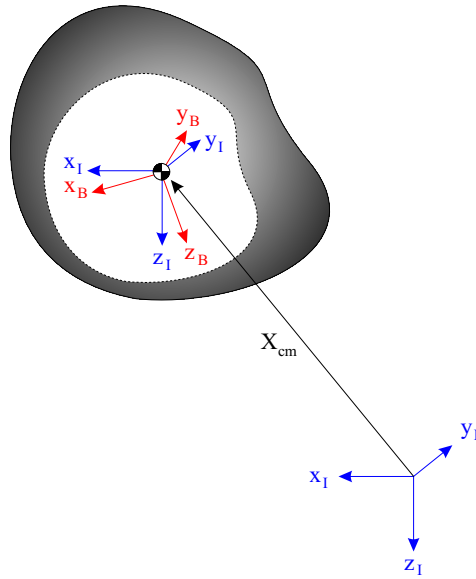


Figure 1: Rigid body coordinate frames.

Recall that we are considering the dynamics of a rigid body. The rigid body dynamic equations, expressed in a reference frame fixed in the body at the center of gravity, are

$$\begin{aligned} m\dot{\mathbf{v}} + \boldsymbol{\omega} \times m\mathbf{v} &= \mathbf{f} \\ \mathbf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{I}\boldsymbol{\omega} &= \mathbf{m}. \end{aligned}$$

Because the  $xz$ -plane of an aircraft is typically a plane of symmetry, it is possible to choose the body coordinate axes in such a way that  $I_{xy} = I_{yz} = 0$ . Writing the equations of motion explicitly, assuming such a choice of body coordinates, we have

$$\begin{aligned} m(\dot{u} + qw - rv) &= X + W_x \\ m(\dot{v} + ru - pw) &= Y + W_y \\ m(\dot{w} + pv - qu) &= Z + W_z \\ I_x\dot{p} - I_{xz}\dot{r} + qr(I_z - I_y) - I_{xz}pq &= L \\ I_y\dot{q} + pr(I_x - I_z) - I_{xz}(p^2 - r^2) &= M \\ I_z\dot{r} - I_{xz}\dot{p} + pq(I_y - I_x) + I_{xz}qr &= N, \end{aligned}$$

where  $X, Y$ , and  $Z$  are the components of the total aerodynamic force (including propulsive forces),  $W_x, W_y$ , and  $W_z$  are the components of weight, and  $L, M$ , and  $N$  are the components of the total aerodynamic moment. These are the rigid body *dynamic equations*, which describe how the forces and moments affect the translational and rotational velocity of the rigid body. We must also develop the *kinematic equations*, which relate translational and rotational velocity to position and attitude.

Suppose that the body merely rotates about its center of mass, without translating, and that the center of mass is located at the origin of the inertial reference frame. (Alternatively, suppose that a reference frame which is aligned with the inertial reference frame is pinned at the center of mass and translates along with the body.) Then the two frames are related by a matrix  $\mathbf{R}_{BI}$  satisfying

$$\mathbf{x}_B = \mathbf{R}_{BI}\mathbf{x}_I, \quad \mathbf{y}_B = \mathbf{R}_{BI}\mathbf{y}_I, \quad \mathbf{z}_B = \mathbf{R}_{BI}\mathbf{z}_I,$$

where  $\mathbf{R}_{\text{BI}}$  is a *proper rotation matrix*, i.e., it is a  $3 \times 3$  rotation matrix which preserves cross-products. Mathematically, we say that

$$\mathbf{R}_{\text{BI}} \in \{ \mathbf{A} \in \mathbb{R}^{3 \times 3} \mid \mathbf{A}^{-1} = \mathbf{A}^T, \det(\mathbf{A}) = +1 \},$$

where the set on the right is the set of all real-valued, proper,  $3 \times 3$  rotation matrices. While a  $3 \times 3$  matrix contains as many as nine independent entries, the condition  $\mathbf{A}^{-1} = \mathbf{A}^T$  can be shown to impose six constraints on the matrix  $\mathbf{R}_{\text{BI}}$ . Thus, at most three independent numbers are required to specify the matrix  $\mathbf{R}_{\text{BI}}$ . (The additional condition that  $\det(\mathbf{A}) = +1$  limits  $\mathbf{R}_{\text{BI}}$  to be a *proper* rotation, i.e., one which preserves right-handed coordinate frames.) There are a great many “attitude parameterizations” available to express the matrix  $\mathbf{R}_{\text{BI}}$ , each with advantages and disadvantages. The one most commonly used in aircraft dynamics is the *XYZ Euler angles*.

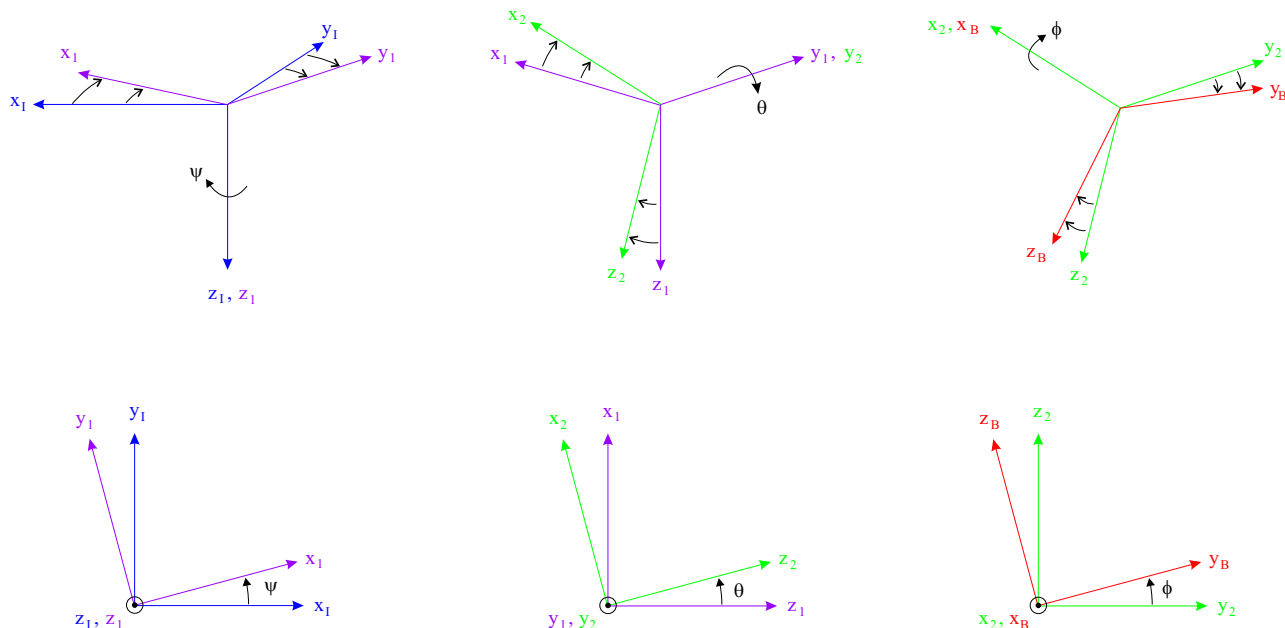


Figure 2: *XYZ Euler angle rotations*.

Suppose that one wishes to transform a vector given in inertial coordinates  $(\mathbf{x}_I, \mathbf{y}_I, \mathbf{z}_I)$  to the rotating body coordinate frame  $(\mathbf{x}_B, \mathbf{y}_B, \mathbf{z}_B)$ . To do so using Euler angles, one first expresses the vector in an intermediate coordinate frame  $(\mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1)$ . This intermediate frame is obtained by rotating the inertial frame in the positive direction about the  $\mathbf{z}_I$  axis through the *yaw angle*  $\psi$ . The transformation is given by the rotation matrix

$$\mathbf{R}_{\text{II}} = \begin{pmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For example, if one wishes to transform the vector  $\mathbf{x}_I = [1, 0, 0]_I^T$  from the inertial frame to the 1-frame, one computes

$$\mathbf{R}_{\text{II}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_I = \begin{pmatrix} \cos \psi \\ -\sin \psi \\ 0 \end{pmatrix}_1 = \cos \psi \mathbf{x}_1 - \sin \psi \mathbf{y}_1.$$

Next, one rotates the intermediate 1-frame in the positive direction about the  $\mathbf{y}_1$  axis through the *pitch angle*  $\theta$ . This transformation defines a new coordinate frame, call it the 2-frame, and is given by the

rotation matrix

$$\mathbf{R}_{21} = \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{pmatrix}.$$

Finally, one rotates the intermediate 2-frame in the positive direction about the  $\mathbf{x}_2$  axis through the *roll angle*  $\phi$ . This transformation is given by the rotation matrix

$$\mathbf{R}_{B2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & \sin \phi \\ 0 & -\sin \phi & \cos \phi \end{pmatrix}.$$

The complete transformation from inertial to body coordinates is given by the composition of the three transformations:

$$\mathbf{R}_{BI} = \mathbf{R}_{B2}\mathbf{R}_{21}\mathbf{R}_{1I} = \begin{pmatrix} \cos \theta \cos \psi & \cos \theta \sin \psi & -\sin \theta \\ \cos \psi \sin \theta \sin \phi - \cos \phi \sin \psi & \cos \phi \cos \psi + \sin \theta \sin \phi \sin \psi & \cos \theta \sin \phi \\ \cos \psi \sin \theta \cos \phi + \sin \phi \sin \psi & -\sin \phi \cos \psi + \sin \theta \cos \phi \sin \psi & \cos \theta \cos \phi \end{pmatrix}. \quad (1)$$

**Example.** Suppose we wish to determine the pitch and roll angle in equilibrium flight from a body-fixed accelerometer measurement. Because an accelerometer measures actual acceleration *plus* a sensed acceleration due to gravity, the measurement in equilibrium flight will be  $-g\mathbf{z}_I$ . Computing

$$\mathbf{a}_{\text{measured}} = \mathbf{R}_{BI}(-g\mathbf{z}_I) = \mathbf{R}_{BI} \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix}_I = -g \begin{pmatrix} -\sin \theta \\ \cos \theta \sin \phi \\ \cos \theta \cos \phi \end{pmatrix}_B,$$

we see that

$$\theta = \arcsin\left(\frac{\mathbf{a}_{\text{measured}_1}}{g}\right) \quad \text{and} \quad \phi = \arcsin\left(-\frac{\mathbf{a}_{\text{measured}_2}}{g \cos \theta}\right),$$

provided  $\cos \theta \neq 0$  (i.e., provided  $\theta \neq \pm\frac{\pi}{2}$ ). Notice that we can not obtain the yaw angle  $\psi$  from a measurement of the direction of gravity. This makes sense because  $\psi$  is measured *about* the direction of gravity. The accelerometer could be rotated through any yaw angle  $\psi$  without affecting the measurement.

Notice also that when  $\theta = \pm\frac{\pi}{2}$ , the axes  $\mathbf{z}_I$  and  $\mathbf{x}_B$  are collinear. In this condition, the yaw angle  $\psi$  is indistinguishable from the roll angle  $\phi$ . The XYZ Euler angle representation is *singular* when  $\theta = \pm\frac{\pi}{2}$ .  $\square$

Because  $\mathbf{R}_{BI}$  is a rotation matrix, its inverse is its transpose. Thus, the inverse transformation from the body to inertial frame is

$$\mathbf{R}_{IB} = \mathbf{R}_{BI}^{-1} = \mathbf{R}_{BI}^T. \quad (2)$$

From here, it is straightforward to express the translational kinematics. If the position of the body's center of mass is given by the inertial coordinate vector  $[x, y, z]_I^T$ , then we have

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}_I = \mathbf{R}_{IB} \begin{pmatrix} u \\ v \\ w \end{pmatrix}_B.$$

Thus, given a translational velocity history in the body frame and the attitude history  $\mathbf{R}_{IB}(t)$ , one may integrate from the initial position in order to determine the body's position in inertial space.

To determine the attitude kinematics (i.e., the relationship between  $\boldsymbol{\omega}$  and the rate of change of  $\mathbf{R}_{IB}$ ), we let the yaw, pitch, and roll angles vary with time and sum their time derivatives  $\dot{\psi}\mathbf{z}_I$ ,  $\dot{\theta}\mathbf{y}_1$ , and  $\dot{\phi}\mathbf{x}_2$  in a

compatible reference frame, say the body reference frame. To this end, we note that

$$\mathbf{R}_{\text{BI}}(\dot{\psi}z_I) = \mathbf{R}_{\text{BI}} \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}_I = \dot{\psi} \begin{pmatrix} -\sin\theta \\ \cos\theta \sin\phi \\ \cos\theta \cos\phi \end{pmatrix}_B,$$

and that

$$\mathbf{R}_{\text{B1}}(\dot{\theta}y_1) = (\mathbf{R}_{\text{B2}}\mathbf{R}_{\text{21}})(\dot{\theta}y_1) = \mathbf{R}_{\text{B2}}(\dot{\theta}y_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} 0 \\ \dot{\theta} \\ 0 \end{pmatrix}_2 = \dot{\theta} \begin{pmatrix} 0 \\ \cos\phi \\ -\sin\phi \end{pmatrix}_B,$$

and finally that

$$\mathbf{R}_{\text{B2}}(\dot{\phi}x_2) = \dot{\phi}x_B = \dot{\phi} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_B.$$

The angular velocity of the body with respect to inertial space, but written in the body frame, is

$$\begin{aligned} \boldsymbol{\omega} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}_B &= \dot{\phi} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_B + \dot{\theta} \begin{pmatrix} 0 \\ \cos\phi \\ -\sin\phi \end{pmatrix}_B + \dot{\psi} \begin{pmatrix} -\sin\theta \\ \cos\theta \sin\phi \\ \cos\theta \cos\phi \end{pmatrix}_B \\ &= \begin{pmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \cos\theta \sin\phi \\ 0 & -\sin\phi & \cos\theta \cos\phi \end{pmatrix} \begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix}. \end{aligned}$$

Note that  $p \neq \dot{\phi}$ ,  $q \neq \dot{\theta}$ , and  $r \neq \dot{\psi}$ , in general!

We define the matrix mapping Euler angle rates to body angular velocity as

$$\mathbf{L}_{\text{BI}} = \begin{pmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \cos\theta \sin\phi \\ 0 & -\sin\phi & \cos\theta \cos\phi \end{pmatrix}.$$

As with the rotation matrix  $\mathbf{R}_{\text{BI}}$ , the subscript ‘‘BI’’ here connotes a map from the inertial frame to the body frame. This is a slight abuse of notation, however, as the vector  $[\dot{\phi}, \dot{\theta}, \dot{\psi}]^T$  is not really a free vector in the same sense as angular rate and angular velocity. The column vector of Euler angle rates is not affiliated with any particular reference frame; it is merely a column of numbers.

The determinant of  $\mathbf{L}_{\text{BI}}$  is

$$\det \mathbf{L}_{\text{BI}} = \cos\theta.$$

Thus, the relationship between body angular velocity and the Euler angle rates may be inverted provided that  $\theta \neq \pm\frac{\pi}{2}$ . Assuming this is the case, we have

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \mathbf{L}_{\text{IB}}\boldsymbol{\omega} \quad \text{where} \quad \mathbf{L}_{\text{IB}} = \mathbf{L}_{\text{BI}}^{-1} = \begin{pmatrix} 1 & \sin\phi \tan\theta & \cos\phi \tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi \sec\theta & \cos\phi \sec\theta \end{pmatrix}. \quad (3)$$

Unlike  $\mathbf{R}_{\text{BI}}$ , the matrix  $\mathbf{L}_{\text{BI}}$  is *not* a rotation matrix. So, for example, its inverse is *not* its transpose!

It should be pointed out that the singularity at  $\theta = \pm\frac{\pi}{2}$  is purely a consequence of the coordinate choice. Certainly there is nothing physical which limits an airplane from obtaining such a pitch angle. The problem can be understood by recalling that, when  $\theta = \pm\frac{\pi}{2}$ , the axes  $z_I$  and  $x_B$  are collinear. In this condition, the yaw angle  $\psi$  is indistinguishable from the roll angle  $\phi$ .

To summarize, the translational and attitude kinematics, with rotations parameterized by Euler angles, are

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \mathbf{R}_{IB} \mathbf{v} \quad (4)$$

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \mathbf{L}_{IB} \boldsymbol{\omega} \quad (5)$$

where  $\mathbf{R}_{IB}$  is given by (2) and (1) and where  $\mathbf{L}_{IB}$  is given by (3). These equations hold provided that  $\theta \neq \pm \frac{\pi}{2}$ .

For convenience, we will lump the primary control parameters into one control vector

$$\mathbf{u} = \begin{pmatrix} \delta a \\ \delta e \\ \delta r \\ \delta T \end{pmatrix}$$

where  $\delta T$  represents a thrust command. Noting that the force of gravity expressed in body coordinates is

$$\mathbf{R}_{BI} \begin{pmatrix} 0 \\ 0 \\ mg \end{pmatrix}_I = mg \begin{pmatrix} -\sin \theta \\ \cos \theta \sin \phi \\ \cos \theta \cos \phi \end{pmatrix}_B,$$

the rigid body dynamic equations are

$$m\dot{\mathbf{v}} + \boldsymbol{\omega} \times m\mathbf{v} = \begin{pmatrix} X(\mathbf{v}, \boldsymbol{\omega}, \mathbf{u}) \\ Y(\mathbf{v}, \boldsymbol{\omega}, \mathbf{u}) \\ Z(\mathbf{v}, \boldsymbol{\omega}, \mathbf{u}) \end{pmatrix} + mg \begin{pmatrix} -\sin \theta \\ \cos \theta \sin \phi \\ \cos \theta \cos \phi \end{pmatrix} \quad (6)$$

$$\mathbf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{I}\boldsymbol{\omega} = \begin{pmatrix} L(\mathbf{v}, \boldsymbol{\omega}, \mathbf{u}) \\ M(\mathbf{v}, \boldsymbol{\omega}, \mathbf{u}) \\ N(\mathbf{v}, \boldsymbol{\omega}, \mathbf{u}) \end{pmatrix}. \quad (7)$$

Equations (4) through (7) completely describe the motion of a rigid airplane. They are twelve first order, nonlinear, time-invariant, ordinary differential equations. In general, these equations are impossible to solve analytically. Often, however, one may learn quite a lot about the behavior of solutions to these equations by studying a simpler set of *approximate* equations which are *linear*, time-invariant. We will next discuss the process of *linearization*.