Lecture 11: Rigid Body Dynamics

In the previous lecture, we derived the following equivalent differential equations governing the translational motion of a system of particles:

$$\dot{\boldsymbol{P}}_{\text{cm}} = \boldsymbol{F} \tag{1}$$

where $\boldsymbol{P}_{\text{cm}} = m\dot{\boldsymbol{X}}_{\text{cm}}$ is the total translational momentum and $\boldsymbol{F}$ is the resultant of all of the *external* forces acting on the system, m is the total system mass, $\boldsymbol{X}_{\text{cm}}$ is the center of mass, and $\boldsymbol{P}_{\text{cm}} = m\dot{\boldsymbol{X}}_{\text{cm}}$ is the total linear moment of the system.

We also derived an expression similar to $\dot{\boldsymbol{P}}_{\text{cm}} = \boldsymbol{F}$ for the rotational motion of the particle system:

$$\dot{\boldsymbol{H}}_{\text{O}} = \boldsymbol{X}_{\text{cm}} \times \boldsymbol{F} + \boldsymbol{M}_{\text{cm}}$$

where

$$\begin{aligned} \boldsymbol{H}_{\text{O}} &= \sum_{i=1}^{N} \boldsymbol{X}_i \times m_i \dot{\boldsymbol{X}}_i \\ &= \boldsymbol{X}_{\text{cm}} \times \underbrace{m\dot{\boldsymbol{X}}_{\text{cm}}}_{\boldsymbol{P}_{\text{cm}}} + \underbrace{\sum_{i=1}^{N} \bar{\boldsymbol{X}}_i \times m_i \dot{\bar{\boldsymbol{X}}}_i}_{\boldsymbol{H}_{\text{cm}}}\,. \end{aligned} \tag{2}$$

is the total angular momentum of the particle system about the origin of the inertial reference frame. The former term is the moment of total linear momentum about the inertial origin and the latter term is the angular momentum about the center of mass.

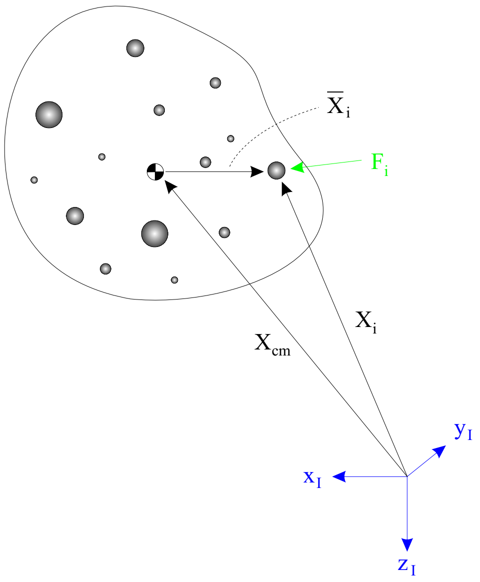

Figure 1: N rigidly constrained particles.

Differentiating (2), and noting that $\dot{\boldsymbol{X}}_{\text{cm}} \parallel \boldsymbol{P}_{\text{cm}}$, gives

$$\dot{\boldsymbol{H}}_{\text{O}} = \boldsymbol{X}_{\text{cm}} \times \dot{\boldsymbol{P}}_{\text{cm}} + \dot{\boldsymbol{H}}_{\text{cm}} \tag{3}$$

But recall from the previous lecture that

$$\dot{\boldsymbol{H}}_{\text{O}} = \boldsymbol{X}_{\text{cm}} \times \boldsymbol{F} + \boldsymbol{M}_{\text{cm}}. \tag{4}$$

Comparing (3) and (4) gives

$$\dot{\boldsymbol{H}}_{\mathrm{cm}} = \boldsymbol{M}_{\mathrm{cm}} \tag{5}$$

We thus obtain an equation for the rotational motion of the system about its center of mass, rather than an equation for its motion about the inertial origin. Equations (1) and (5) describe the aggregate translational and rotational dynamics of the system of particles.

Now, suppose that the collection of particles forms a rigid body. That is, suppose that no particle moves relative to any other particle in the system. In this case, we find that

$$\boldsymbol{H}_{\mathrm{cm}} = \sum_{i=1}^{N} \bar{\boldsymbol{X}}_i \times m_i(\boldsymbol{\omega}_{\mathrm{I}} \times \bar{\boldsymbol{X}}_i),$$

where $\boldsymbol{\omega}_{\mathrm{I}}$ is the angular velocity of the body with respect to inertial space, written in the inertial frame. (The apparent paradox – that "no particle moves relative to any other" while, at the same time, $\dot{\bar{\boldsymbol{X}}}_i = \boldsymbol{\omega}_{\mathrm{I}} \times \bar{\boldsymbol{X}}_i$ – is a consequence of the choice of reference frame in which the velocity is computed. More in a moment.) Assume that each particle has the same infinitesimal mass $m_i = dm$. Then, in the limit $N \to \infty$, the summation becomes an integral:

$$\boldsymbol{H}_{\mathrm{cm}} = -\int_{\mathrm{V}} \bar{\boldsymbol{X}} \times (\bar{\boldsymbol{X}} \times \boldsymbol{\omega}_{\mathrm{I}})dm$$

where the vector $\bar{\boldsymbol{X}} = \bar{X}\boldsymbol{x}_{\mathrm{B}} + \bar{Y}\boldsymbol{y}_{\mathrm{B}} + \bar{Z}\boldsymbol{z}_{\mathrm{B}}$ denotes the location of a point in the rigid body relative to the center of mass and expressed in the inertial frame. As an exercise, you may check that

$$-\int_{\mathrm{V}} \bar{\boldsymbol{X}} \times (\bar{\boldsymbol{X}} \times \boldsymbol{\omega}_{\mathrm{I}})dm = \boldsymbol{I}_{\mathrm{I}}\boldsymbol{\omega}_{\mathrm{I}} \qquad \text{where} \qquad \boldsymbol{I}_{\mathrm{I}} = \int_{\mathrm{V}} \begin{pmatrix} \bar{Y}^2 + \bar{Z}^2 & -\bar{X}\bar{Y} & -\bar{X}\bar{Z} \\ -\bar{X}\bar{Y} & \bar{X}^2 + \bar{Z}^2 & -\bar{Y}\bar{Z} \\ -\bar{X}\bar{Z} & -\bar{Y}\bar{Z} & \bar{X}^2 + \bar{Y}^2 \end{pmatrix} dm.$$

We now have an explicit expression for $\boldsymbol{H}_{\mathrm{cm}}$ in terms of the body's mass distribution and the body angular rate $\boldsymbol{\omega}_{\mathrm{I}}$ which we may use in equation (5). The problem is that the inertia tensor $\boldsymbol{I}_{\mathrm{I}}$ that defines $\boldsymbol{H}_{\mathrm{cm}}$ varies with time. Because the body rotates with respect to inertial space, the components of $\bar{\boldsymbol{X}}$ vary with time and so does $\boldsymbol{I}_{\mathrm{I}}$. (We'll see how in just a minute.)

It turns out to be much easier to express the equations of motion in a reference frame that is fixed in the body. In this case, the inertia tensor is constant. Also, aerodynamic forces and moments are more naturally expressed with respect to the body than with respect to inertial space.

Let us re-express the preceding developments in terms of a coordinate frame $(\boldsymbol{x}_{\mathrm{B}}, \boldsymbol{y}_{\mathrm{B}}, \boldsymbol{z}_{\mathrm{B}})$ which is fixed to the rigid body at its center of mass. To start with, we define the angular momentum $\boldsymbol{h}_{\mathrm{cm}}$ about the center of mass and *expressed in the body frame*, as

$$\boldsymbol{h}_{\mathrm{cm}} = -\int_{\mathrm{V}} \bar{\boldsymbol{x}} \times (\bar{\boldsymbol{x}} \times \boldsymbol{\omega}_{\mathrm{B}})dm.$$

where the body frame vector $\bar{\boldsymbol{x}} = \bar{x}\boldsymbol{x}_{\mathrm{B}} + \bar{y}\boldsymbol{y}_{\mathrm{B}} + \bar{z}\boldsymbol{z}_{\mathrm{B}}$ denotes the location of a point in the rigid body relative to the center of mass and expressed in the body frame and $\boldsymbol{\omega}_{\mathrm{B}} = [p,\ q,\ r]^T$ is the angular velocity of the body with respect to inertial space expressed in the body frame. We thus have

$$-\int_{\mathrm{V}} \bar{\boldsymbol{x}} \times (\bar{\boldsymbol{x}} \times \boldsymbol{\omega}_{\mathrm{B}})dm = \boldsymbol{I}_{\mathrm{B}}\boldsymbol{\omega}_{\mathrm{B}} \qquad \text{where} \qquad \boldsymbol{I}_{\mathrm{B}} = \int_{\mathrm{V}} \begin{pmatrix} \bar{y}^2 + \bar{z}^2 & -\bar{x}\bar{y} & -\bar{x}\bar{z} \\ -\bar{x}\bar{y} & \bar{x}^2 + \bar{z}^2 & -\bar{y}\bar{z} \\ -\bar{x}\bar{z} & -\bar{y}\bar{z} & \bar{x}^2 + \bar{y}^2 \end{pmatrix} dm. \tag{6}$$

Because the reference frame in which these integrals are expressed moves along with the body, every component of $\boldsymbol{I}_{\mathrm{B}}$ is *constant.* We define the moments and products of inertia according to the relationship

$$\begin{pmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{xy} & I_y & -I_{yz} \\ -I_{xz} & -I_{yz} & I_z \end{pmatrix} := \int_{\mathrm{V}} \begin{pmatrix} \bar{y}^2 + \bar{z}^2 & -\bar{x}\bar{y} & -\bar{x}\bar{z} \\ -\bar{x}\bar{y} & \bar{x}^2 + \bar{z}^2 & -\bar{y}\bar{z} \\ -\bar{x}\bar{z} & -\bar{y}\bar{z} & \bar{x}^2 + \bar{y}^2 \end{pmatrix} dm$$

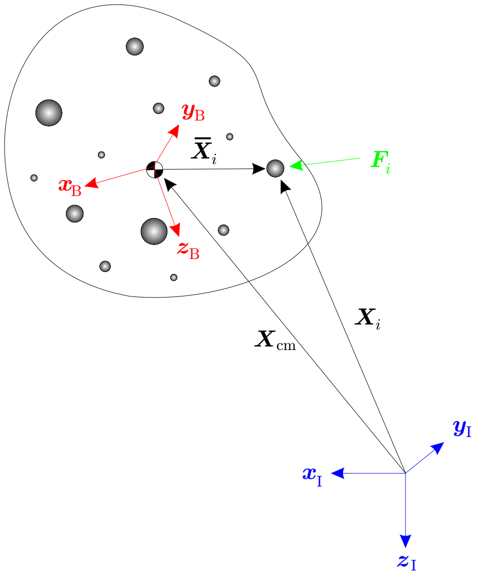

Figure 2: Inertial and body-fixed coordinate frames.

Comment: Suppose that the proper rotation matrix $\boldsymbol{R}_{\mathrm{IB}}(t)$ transforms free vectors expressed in the body reference frame into the inertial reference frame. Then $\boldsymbol{\omega}_{\mathrm{I}} = \boldsymbol{R}_{\mathrm{IB}}\boldsymbol{\omega}_{\mathrm{B}}$ and $\boldsymbol{H}_{\mathrm{cm}} = \boldsymbol{R}_{\mathrm{IB}}\boldsymbol{h}_{\mathrm{cm}}$, so that

$$\begin{aligned}\boldsymbol{I}_{\mathrm{I}}\boldsymbol{\omega}_{\mathrm{I}} &= \boldsymbol{R}_{\mathrm{IB}}\left(\boldsymbol{I}_{\mathrm{B}}\boldsymbol{\omega}_{\mathrm{B}}\right)\\ &= \boldsymbol{R}_{\mathrm{IB}}\boldsymbol{I}_{\mathrm{B}}\boldsymbol{R}_{\mathrm{IB}}^{-1}\boldsymbol{\omega}_{\mathrm{I}}.\end{aligned}$$

Since this equation holds for *any* angular velocity vector $\boldsymbol{\omega}_{\mathrm{I}}$, we conclude that

$$\boldsymbol{I}_{\mathrm{I}} = \boldsymbol{R}_{\mathrm{IB}}\boldsymbol{I}_{\mathrm{B}}\boldsymbol{R}_{\mathrm{IB}}^{-1}.$$

The rotation matrix $\boldsymbol{R}_{\mathrm{IB}}$ is a function of time because the body is rotating. Thus, while the matrix $\boldsymbol{I}_{\mathrm{B}}$ is a matrix of constants, the matrix $\boldsymbol{I}_{\mathrm{I}}$ is time-varying; at any instant, it depends on the orientation of the body-fixed frame with respect to inertial space. ☐

To summarize, we have $\boldsymbol{h}_{\mathrm{cm}} = \boldsymbol{I}_{\mathrm{B}}\boldsymbol{\omega}_{\mathrm{B}}$, where $\boldsymbol{I}_{\mathrm{B}}$ is a constant matrix. If we define $\boldsymbol{m}_{\mathrm{cm}} = \boldsymbol{R}_{\mathrm{BI}}\boldsymbol{M}_{\mathrm{cm}}$ to be the total moment about the center of mass, expressed in the body reference frame, then we have only to relate the rate of change of $\boldsymbol{h}_{\mathrm{cm}}$ to $\boldsymbol{m}_{\mathrm{cm}}$. This is not quite as simple as it might appear: $\dot{\boldsymbol{h}}_{\mathrm{cm}} \neq \boldsymbol{m}_{\mathrm{cm}}$! While having a constant inertia matrix does simplify the equations of motion, a new complication arises due to the rotating body-fixed reference frame.

Rotating reference frames. The time rate of change of a vector written with respect to an inertially fixed frame is simply the derivative of the vector's components. The time rate of change of a vector written with respect to a rotating coordinate frame, however, depends also on the rate of rotation of the coordinate frame. Let $\boldsymbol{a}(t)$ be a vector expressed with respect to a rotating reference frame. Let $\boldsymbol{A}(t) = \boldsymbol{R}_{\mathrm{IB}}(t)\boldsymbol{a}(t)$ represent the same vector, but expressed in an inertial frame. Then

$$\begin{aligned}\left.\frac{d}{dt}\right|_{\mathrm{Inertial}}(\boldsymbol{a}) &= \left.\frac{d}{dt}\right|_{\mathrm{Body}}(\boldsymbol{a}) + \boldsymbol{\omega}_{\mathrm{B}} \times \boldsymbol{a}\\ &= \dot{\boldsymbol{a}} + \boldsymbol{\omega}_{\mathrm{B}} \times \boldsymbol{a} \qquad (7)\end{aligned}$$

where

$$\left.\frac{d}{dt}\right|_{\mathrm{Inertial}}(\boldsymbol{a}) = \boldsymbol{R}_{\mathrm{BI}}\dot{\boldsymbol{A}}. \qquad (8)$$

(See any undergraduate dynamics textbook for a proof.) On the left hand side of equation (7), the time derivative is taken with respect to inertial space; on the right hand side, the time derivative is taken with respect to the rotating coordinate frame. On both sides of the equation, vectors are expressed in the body frame. Notice the important distinction between the phrases "*with respect to*" and "*expressed in*." Although they are taken *with respect to* different reference frames, the time derivatives on either side of (8) are *expressed in* the same frame. If one desired, one could re-express the vectors in the inertial frame by premultiplying both sides by $\boldsymbol{R}_{IB} = \boldsymbol{R}_{BI}^{-1}$. In that case, we would have

$$\begin{aligned}\dot{\boldsymbol{A}} &= \boldsymbol{R}_{IB}\left(\dot{\boldsymbol{a}} + \boldsymbol{\omega}_{\mathrm{B}} \times \boldsymbol{a}\right) \\ &= \boldsymbol{R}_{IB}\dot{\boldsymbol{a}} + (\boldsymbol{R}_{IB}\boldsymbol{\omega}_{\mathrm{B}}) \times (\boldsymbol{R}_{IB}\boldsymbol{a}) \\ &= \boldsymbol{R}_{IB}\dot{\boldsymbol{a}} + \boldsymbol{\omega}_{\mathrm{I}} \times \boldsymbol{A}.\end{aligned}$$

(Notice that left multiplication by a rotation matrix distributes over the cross-product; this is *not* true for general, non-rotation matrices.)

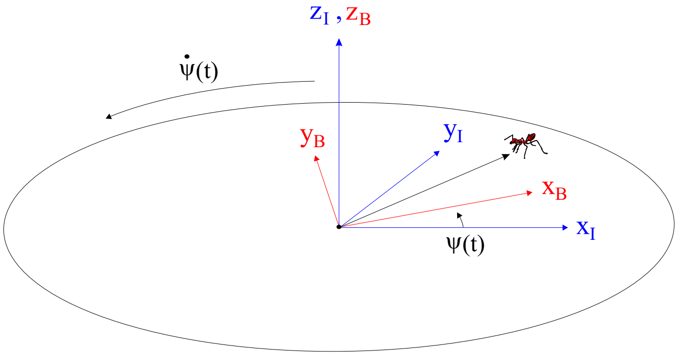

Figure 3: An ant on a turntable.

Example: An ant on a turntable. Think of an ant walking on top of a spinning turntable. The angle ψ is the counterclockwise angle from the inertial $\boldsymbol{x}_{\mathrm{I}}$ axis to the body-fixed $\boldsymbol{x}_{\mathrm{B}}$ axis, as shown in Figure 3. The transformation from the body frame to the inertial frame is given by the rotation matrix

$$\boldsymbol{R}_{\mathrm{IB}}(t) = \begin{pmatrix} \cos\psi(t) & -\sin\psi(t) & 0 \\ \sin\psi(t) & \cos\psi(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let us check this assertion by mapping the unit vector $\boldsymbol{x}_{\mathrm{B}}$ into the inertial frame:

$$\boldsymbol{R}_{\mathrm{IB}}(t)\boldsymbol{x}_{\mathrm{B}} = \begin{pmatrix} \cos\psi(t) & -\sin\psi(t) & 0 \\ \sin\psi(t) & \cos\psi(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{\mathrm{B}} = \begin{pmatrix} \cos\psi(t) \\ \sin\psi(t) \\ 0 \end{pmatrix}_{\mathrm{I}}$$

As a simple check, refer to Figure 3 and notice that, for small, positive angles ψ, the vector $\boldsymbol{x}_{\mathrm{B}}$, re-expressed in the inertial frame, should have a relatively large positive component ($\cos\psi$) in the $\boldsymbol{x}_{\mathrm{I}}$ direction and a small positive component ($\sin\psi$) in the $\boldsymbol{y}_{\mathrm{I}}$ direction.

The inverse transformation, i.e., the map from the inertial frame to the body frame, is

$$\boldsymbol{R}_{\mathrm{BI}}(t) = \boldsymbol{R}_{\mathrm{IB}}(t)^{-1} = \boldsymbol{R}_{\mathrm{IB}}(t)^{T} = \begin{pmatrix} \cos\psi(t) & \sin\psi(t) & 0 \\ -\sin\psi(t) & \cos\psi(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The turntable spins with inertial angular velocity

$$\boldsymbol{\omega}_{\mathrm{I}} = \frac{d}{dt}\left(\psi(t)\boldsymbol{z}_{\mathrm{I}}\right) = \dot{\psi}\boldsymbol{z}_{\mathrm{I}} = \begin{pmatrix} 0 \\ 0 \\ \dot{\psi}(t) \end{pmatrix}_{\mathrm{I}}.$$

We may convert the angular velocity from the inertial frame to the body frame by writing

$$\boldsymbol{\omega}_{\rm B} = \boldsymbol{R}_{\rm BI}(t)\boldsymbol{\omega}_{\rm I} = \begin{pmatrix} \cos\psi(t) & \sin\psi(t) & 0 \\ -\sin\psi(t) & \cos\psi(t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}_{\rm I} = \begin{pmatrix} 0 \\ 0 \\ \dot{\psi}(t) \end{pmatrix}_{\rm B} = \dot{\psi}(t)\boldsymbol{z}_{\rm B}.$$

For this simple example of planar rotation, the two vectors $\boldsymbol{\omega}_{\rm I}$ and $\boldsymbol{\omega}_{\rm B}$ are numerically equal. This is not true for general, three-dimensional rotations.

Suppose that, at a given instant t, the ant is located by the body frame vector

$$\boldsymbol{x}_{\rm ant} = \begin{pmatrix} x(t) \\ y(t) \\ 0 \end{pmatrix}_{\rm B}.$$

In the inertial frame, we have

$$\begin{aligned} \boldsymbol{X}_{\rm ant} = \begin{pmatrix} X(t) \\ Y(t) \\ Z(t) \end{pmatrix}_{\rm I} = \boldsymbol{R}_{\rm IB}\boldsymbol{x}_{\rm ant} &= \begin{pmatrix} \cos\psi(t) & -\sin\psi(t) & 0 \\ \sin\psi(t) & \cos\psi(t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \\ 0 \end{pmatrix}_{\rm B} \\ &= \begin{pmatrix} x(t)\cos\psi(t) - y(t)\sin\psi(t) \\ x(t)\sin\psi(t) + y(t)\cos\psi(t) \\ 0 \end{pmatrix}_{\rm I}. \end{aligned} \tag{9}$$

The velocity of the ant with respect to inertial space, but written in the body frame, can be obtained from (7) as

$$\left.\frac{d}{dt}\right|_{\rm Inertial} (\boldsymbol{x}_{\rm ant}) = \dot{\boldsymbol{x}}_{\rm ant} + \boldsymbol{\omega}_{\rm B} \times \boldsymbol{x}_{\rm ant}$$

or

$$\boldsymbol{R}_{\rm BI}\dot{\boldsymbol{X}}_{\rm ant} = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ 0 \end{pmatrix}_{\rm B} + \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}_{\rm B} \times \begin{pmatrix} x(t) \\ y(t) \\ 0 \end{pmatrix}_{\rm B} = \begin{pmatrix} \dot{x}(t) - y(t)\dot{\psi} \\ \dot{y}(t) + x(t)\dot{\psi} \\ 0 \end{pmatrix}_{\rm B}.$$

As an exercise, you might verify this result by computing the total time-derivative of $\boldsymbol{X}_{\rm ant}$ directly from (9) and then premultiplying by $\boldsymbol{R}_{\rm BI}$.

If we smash the ant, so that it becomes part of the rigid body, then the relative velocity $\dot{\boldsymbol{x}}_{\rm ant}$ will be zero and the inertial velocity is due entirely to the angular velocity of the turntable. □

Rigid body dynamic equations in body coordinates. The complete dynamic equations for a rigid body, written in the inertial frame, are

$$\begin{aligned} \dot{\boldsymbol{P}}_{\rm cm} &= \boldsymbol{F} \\ \dot{\boldsymbol{H}}_{\rm cm} &= \boldsymbol{M}_{\rm cm} \end{aligned} \tag{10}$$

These two vector equations seem harmless enough, but recall that $\boldsymbol{H}_{\rm cm} = \boldsymbol{I}_{\rm I}\boldsymbol{\omega}_{\rm I}$, where $\boldsymbol{I}_{\rm I}$ is the (time-dependent) inertia matrix computed with respect to the inertial frame. The left-hand side of the latter equation is thus fairly complicated. Because of this, and because aerodynamic forces on an airplane are more easily expressed in a body-fixed coordinate frame, it is more convenient to express the dynamic equations in the body frame. To do so, we use the identity (8). Let $\boldsymbol{V}_{\rm cm} = \dot{\boldsymbol{X}}_{\rm cm}$ represent the inertial velocity of a given rigid body's center of mass. Let

$$\boldsymbol{v}_{\rm cm} = \boldsymbol{R}_{\rm BI}\boldsymbol{V}_{\rm cm} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}_{\rm B}$$

represent the same translational velocity, but written in a frame fixed at the body's center of mass. With this definition, we may write the translational momentum and the angular momentum about the mass center in the body frame as

$$\begin{aligned} \boldsymbol{p}_{\rm cm} &= \boldsymbol{R}_{\rm BI}\boldsymbol{P}_{\rm cm} &&= \boldsymbol{R}_{\rm BI}\left(m\boldsymbol{V}_{\rm cm}\right) &&= m\boldsymbol{v}_{\rm cm} \\ \boldsymbol{h}_{\rm cm} &= \boldsymbol{R}_{\rm BI}\boldsymbol{H}_{\rm cm} &&= \boldsymbol{R}_{\rm BI}\left(\boldsymbol{I}_{\rm I}\boldsymbol{\omega}_{\rm I}\right) &&= \boldsymbol{I}_{\rm B}\boldsymbol{\omega}_{\rm B}. \end{aligned} \tag{11}$$

(In the latter equation, we have substituted $\boldsymbol{\omega}_{\rm B} = \boldsymbol{R}_{\rm IB}^{-1}\boldsymbol{\omega}_{\rm I}$ and used the observation that $\boldsymbol{I}_{\rm B} = \boldsymbol{R}_{\rm BI}\boldsymbol{I}_{\rm I}\boldsymbol{R}_{\rm BI}^{-1}$.) Applying the identity (7) to compute the derivatives of $\boldsymbol{g}_{\rm cm}$ and $\boldsymbol{h}_{\rm cm}$ with respect to inertial space gives

$$\begin{aligned} \left.\frac{d}{dt}\right|_{\rm Inertial}\left(\boldsymbol{p}_{\rm cm}\right) &= \boldsymbol{R}_{\rm BI}\dot{\boldsymbol{P}}_{\rm cm} &&= \dot{\boldsymbol{p}}_{\rm cm} + \boldsymbol{\omega}_{\rm B} \times \boldsymbol{p}_{\rm cm} \\ \left.\frac{d}{dt}\right|_{\rm Inertial}\left(\boldsymbol{h}_{\rm cm}\right) &= \boldsymbol{R}_{\rm BI}\dot{\boldsymbol{H}}_{\rm cm} &&= \dot{\boldsymbol{h}}_{\rm cm} + \boldsymbol{\omega}_{\rm B} \times \boldsymbol{h}_{\rm cm}. \end{aligned}$$

Substituting $\dot{\boldsymbol{P}}_{\rm cm} = \boldsymbol{F}$ and $\dot{\boldsymbol{H}}_{\rm cm} = \boldsymbol{M}_{\rm cm}$ from (10), as well as the definitions of $\boldsymbol{p}_{\rm cm}$ and $\boldsymbol{h}_{\rm cm}$ from (11), gives

$$\begin{aligned} \boldsymbol{R}_{\rm BI}\boldsymbol{F} &= m\dot{\boldsymbol{v}}_{\rm cm} + \boldsymbol{\omega}_{\rm B} \times m\boldsymbol{v}_{\rm cm} \\ \boldsymbol{R}_{\rm BI}\boldsymbol{M}_{\rm cm} &= \boldsymbol{I}_{\rm B}\dot{\boldsymbol{\omega}}_{\rm B} + \boldsymbol{\omega}_{\rm B} \times \boldsymbol{I}_{\rm B}\boldsymbol{\omega}_{\rm B}. \end{aligned}$$

Defining the body frame expressions for force and moment, $\boldsymbol{f} = \boldsymbol{R}_{\rm BI}\boldsymbol{F}$ and $\boldsymbol{m}_{\rm cm} = \boldsymbol{R}_{\rm BI}\boldsymbol{M}_{\rm cm}$, we finally obtain

$$\begin{aligned} m\dot{\boldsymbol{v}}_{\rm cm} + \omega_{\rm B} \times m\boldsymbol{v}_{\rm cm} &= \boldsymbol{f} \\ \boldsymbol{I}_{\rm B}\dot{\boldsymbol{\omega}}_{\rm B} + \boldsymbol{\omega}_{\rm B} \times \boldsymbol{I}_{\rm B}\boldsymbol{\omega}_{\rm B} &= \boldsymbol{m}_{\rm cm}. \end{aligned}$$

As we will almost always express equations in a body frame fixed at the center of mass, we will typically omit the related subscripts:

$$\boxed{\begin{aligned} m\dot{\boldsymbol{v}} + \boldsymbol{\omega} \times m\boldsymbol{v} &= \boldsymbol{f} \\ \boldsymbol{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \boldsymbol{I}\boldsymbol{\omega} &= \boldsymbol{m}. \end{aligned}}$$

Because the xz-plane of an aircraft is typically a plane of symmetry, it is possible to choose the body reference frame such that $I_{xy} = I_{yz} = 0$.[1] Writing the equations of motion explicitly, assuming such a choice of body reference frame, we have

$$\boxed{\begin{aligned} m(\dot{u} + qw - rv) &= X + W_x \\ m(\dot{v} + ru - pw) &= Y + W_y \\ m(\dot{w} + pv - qu) &= Z + W_z \\ I_x\dot{p} - I_{xz}\dot{r} + qr(I_z - I_y) - I_{xz}pq &= L \\ I_y\dot{q} + pr(I_x - I_z) - I_{xz}(p^2 - r^2) &= M \\ I_z\dot{r} - I_{xz}\dot{p} + pq(I_y - I_x) + I_{xz}qr &= N. \end{aligned}}$$

These six first order, nonlinear, ordinary differential equations (ODEs) describe the motion of a rigid body whose body xz-plane is a plane of symmetry. The force terms W_x, W_y, and W_z are the components of

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[1] In fact, one could choose a reference frame whose axes are aligned with the principal axes of inertia. In this case, all cross products of inertia would vanish. Such a choice, however, makes the expression of aerodynamic forces and moments less convenient.

the body weight in the body reference frame. The remaining force and moment components on the right arise due to aerodynamic effects. These equations are impossible to solve analytically, except in very special cases. One may certainly solve the equations numerically, for given forcing and initial conditions, but pencil-and-paper analysis is challenging. We therefore must develop a simpler set of *approximate* equations. This will be a topic for a later class.

The equations above are the *dynamic equations*, which describe how external forces and moments affect the translational and rotational velocity of the rigid body. In the next lecture, we will derive six more first order ODEs which relate translational and rotational velocity to position and attitude. These equations are known as the rigid body *kinematic equations*.