

# Lecture 10: Review of Particle Dynamics

**Static stability versus dynamic stability.** At this point, we have discussed the notion of static stability for longitudinal and lateral-directional motions. Static stability relates to an airplane's *initial tendency* in response to a perturbation from equilibrium flight – if this initial tendency is to return to equilibrium flight, then we say the motion is statically stable. Dynamic stability concerns not only the short term, but the long term as well. Dynamic stability requires that the system truly returns to the equilibrium after a small perturbation. The advantage of considering static stability over dynamic stability is that static stability can be assessed without solving, or even deriving, the complete equations of motion.

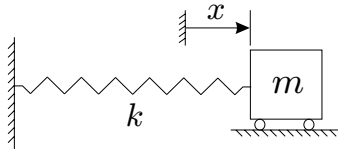


Figure 1: A mass-spring system.

As an example, consider a simple mass-spring

$$m\ddot{x} = -kx,$$

where  $x$  is measured from the equilibrium position. We can solve this simple linear, time-invariant ODE explicitly, but that is unnecessary to determine static stability of the equilibrium. Simply recognize that, if the system is perturbed to a new state  $(x, \dot{x}) = (x_0, 0)$  where  $x_0 \neq 0$ , then the system will initially tend to return to the equilibrium provided  $k > 0$ :

$$\ddot{x}(0) = -\frac{k}{m}x(0) \quad \begin{cases} < 0 & x_0 > 0 \\ > 0 & x_0 < 0 \end{cases}$$

If the mass is pulled to some initial point  $x_0$ , held at rest, and then released, it will initially accelerate back toward the equilibrium. If  $k < 0$ , on the other hand, the perturbed mass will initially tend to move away from the equilibrium. Thus, positive spring stiffness is necessary and sufficient for static stability.

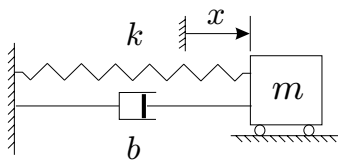


Figure 2: A mass-spring-damper system.

Next consider a mass-spring-damper

$$m\ddot{x} = -b\dot{x} - kx, \tag{1}$$

where we assume that  $m > 0$  and  $k > 0$ , but we allow the possibility that  $b < 0$ . Once again, suppose that the system is perturbed to a new state  $(x, \dot{x}) = (x_0, 0)$  where  $x_0 \neq 0$ . Noting that

$$\begin{aligned} \ddot{x}(0) &= -\frac{b}{m}\dot{x}(0) - \frac{k}{m}x(0) \\ &= -\frac{k}{m}x(0), \end{aligned}$$

as before, we see that the system is, once again, statically stable. The perturbed mass will *initially* move in the direction of the equilibrium regardless of the value of  $b$ .

Assuming that  $-1 \leq \zeta \leq 1$ , the general solution to equation (1), with the given perturbed initial condition, is

$$\begin{aligned} x(t) &= x_0 e^{-\zeta \omega_n t} (\cos \omega_d t + \omega_d \sin \omega_d t) \\ &= \frac{x_0 e^{-\zeta \omega_n t}}{\sqrt{1 - \zeta^2}} \cos(\omega_d t - \phi) \end{aligned}$$

where

$$\omega_n = \sqrt{\frac{k}{m}}, \quad \zeta = \frac{b}{2\sqrt{km}}, \quad \omega_d = \omega_n \sqrt{1 - \zeta^2}, \quad \text{and} \quad \phi = \arctan \frac{\zeta}{\sqrt{1 - \zeta^2}}.$$

If  $b < 0$ , then  $-1 < \zeta < 0$  and the exponential envelope, and the sinusoid which it bounds, diverges as  $t \rightarrow \infty$ . Even though the equilibrium is statically stable, it is (dynamically) unstable. Clearly, some additional condition(s) besides  $k > 0$  must be imposed to ensure dynamic stability. For this simple example, the additional requirement  $b > 0$  is sufficient to ensure dynamic stability, but higher dimensional systems such as airplanes require more careful analysis.

The essential problem in assessing dynamic stability is to infer the qualitative behavior of the system, in response to small perturbations, over long time intervals. That is, one must understand the nature of *solutions* to the governing differential equations, not just their initial tendency. As a first step toward understanding dynamic stability of equilibrium flight, we must develop the dynamic equations for an airplane. We will start at the very beginning.

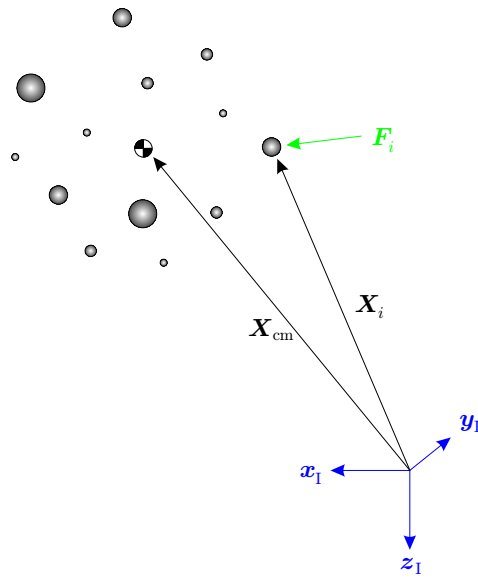


Figure 3: System of particles.

**Particle dynamics: Translational Motion.** Consider a collection of  $N$  point masses, as shown in Figure 3. Suppose that the  $i^{\text{th}}$  point mass has mass  $m_i$  and that its location in inertial space is given by the vector  $\mathbf{X}_i$ . (We will use capital letters to denote vectors expressed in the inertial reference frame.) Suppose that a net external force  $\mathbf{F}_i$  acts on the  $i^{\text{th}}$  point mass. Also, suppose there is a force of interaction  $\mathbf{F}_{ij}$  exerted by the  $j^{\text{th}}$  particle on the  $i^{\text{th}}$ . Newton's second law of motion applied to the  $i^{\text{th}}$  particle gives

$$m_i \ddot{\mathbf{X}}_i = \mathbf{F}_i + \sum_{j=1}^N \mathbf{F}_{ij}.$$

Summing over the  $N$  particles gives

$$\frac{d^2}{dt^2} \sum_{i=1}^N m_i \mathbf{X}_i = \sum_{i=1}^N \left( \mathbf{F}_i + \sum_{j=1}^N \mathbf{F}_{ij} \right).$$

Now, assume that the interaction forces are equal and opposite:  $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$ . (Note that this implies  $F_{ii} = 0$ ; the  $i^{\text{th}}$  point mass exerts no force on itself.) All of the forces of interaction cancel in the summation, leaving

$$\frac{d^2}{dt^2} \sum_{i=1}^N m_i \mathbf{X}_i = \sum_{i=1}^N \mathbf{F}_i. \quad (2)$$

The *center of mass* of the collection of particles is defined as

$$\mathbf{X}_{\text{cm}} = \frac{1}{m} \left( \sum_{i=1}^N m_i \mathbf{X}_i \right) \quad \text{where} \quad m = \sum_{i=1}^N m_i.$$

Referring to (2), we see that

$$\frac{d^2}{dt^2} (m \mathbf{X}_{\text{cm}}) = \sum_{i=1}^n \mathbf{F}_i.$$

Recall that two force systems are equivalent provided the net magnitude and direction of the forces is equal. (This is not true for moment systems, where the point of action of each force is also important.) We may therefore define an equivalent force

$$\mathbf{F} = \sum_{i=1}^n \mathbf{F}_i$$

which acts at the point  $\mathbf{X}_{\text{cm}}$ . Thus, we may write

$$m \ddot{\mathbf{X}}_{\text{cm}} = \mathbf{F}. \quad (3)$$

Alternatively, suppose that we define the total system linear momentum

$$\mathbf{P} = \frac{d}{dt} \left( \sum_{i=1}^N m_i \mathbf{X}_i \right).$$

Note that, by definition,

$$\mathbf{P} = \frac{d}{dt} (m \mathbf{X}_{\text{cm}}),$$

so the combined linear momentum of the system of particles is equivalent to the momentum of a single particle of mass  $m$  located at the center of mass. We therefore define  $\mathbf{P}_{\text{cm}} = \mathbf{P}$ , that is,

$$\mathbf{P}_{\text{cm}} = m \dot{\mathbf{X}}_{\text{cm}}.$$

In terms of the linear momentum, Newton's second law of motion (3) for the system is

$$\dot{\mathbf{P}}_{\text{cm}} = \mathbf{F}. \quad (4)$$

In summary, the effect of the  $N$  point forces  $\mathbf{F}_i$  acting on the each of the  $N$  particles is equivalent to the effect of a single force  $\mathbf{F}$  on a single point mass  $m$  located at  $\mathbf{X}_{\text{cm}}$ . If one were to solve the dynamic equations for each particle separately and compute the resulting center of mass location  $\mathbf{X}_{\text{cm}}(t)$ , it would be identical to the solution to equation (3). The equivalence holds even as the various particles move

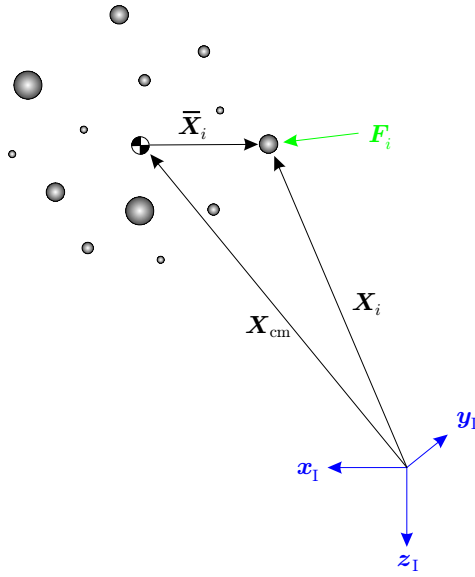


Figure 4: System of particles.

toward each other, away from each other, collide elastically or plastically, and so on. This is an important, fundamental observation about the motion of systems of particles.

**Particle Dynamics: Rotational Motion.** To determine the rotational motion of the particle system, we must go back and consider the moment generated by each point force  $\mathbf{F}_i$ . The moment of translational momentum (the *angular* momentum) of the  $i^{\text{th}}$  particle about the inertial frame origin is

$$\mathbf{H}_i = \mathbf{X}_i \times m_i \dot{\mathbf{X}}_i.$$

Summing the contributions  $\mathbf{H}_i$  over the  $N$  particles gives the total system momentum about the inertial frame origin:

$$\mathbf{H}_O = \sum_{i=1}^N \mathbf{X}_i \times m_i \dot{\mathbf{X}}_i$$

Let  $\bar{\mathbf{X}}_i$  denote the position of the  $i^{\text{th}}$  point mass with respect to the center of mass, expressed in the inertial frame. Then  $\mathbf{X}_i = \mathbf{X}_{\text{cm}} + \bar{\mathbf{X}}_i$  and we have

$$\begin{aligned} \mathbf{H}_O &= \sum_{i=1}^N \left\{ (\mathbf{X}_{\text{cm}} + \bar{\mathbf{X}}_i) \times m_i (\dot{\mathbf{X}}_{\text{cm}} + \dot{\bar{\mathbf{X}}}_i) \right\} \\ &= \sum_{i=1}^N \left( \mathbf{X}_{\text{cm}} \times m_i \dot{\mathbf{X}}_{\text{cm}} + \mathbf{X}_{\text{cm}} \times \frac{d}{dt} (m_i \bar{\mathbf{X}}_i) + m_i \bar{\mathbf{X}}_i \times \dot{\mathbf{X}}_{\text{cm}} + \bar{\mathbf{X}}_i \times m_i \dot{\bar{\mathbf{X}}}_i \right) \\ &= \mathbf{X}_{\text{cm}} \times \left( \sum_{i=1}^N m_i \right) \dot{\mathbf{X}}_{\text{cm}} + \mathbf{X}_{\text{cm}} \times \frac{d}{dt} \left( \sum_{i=1}^N m_i \bar{\mathbf{X}}_i \right) + \left( \sum_{i=1}^N m_i \bar{\mathbf{X}}_i \right) \times \dot{\mathbf{X}}_{\text{cm}} + \sum_{i=1}^N \left( \bar{\mathbf{X}}_i \times m_i \dot{\bar{\mathbf{X}}}_i \right) \end{aligned}$$

The middle two terms vanish by definition of the center of mass. Thus, the total angular momentum of the system about the origin of the inertial reference frame is the sum of the moment of translational momentum (as if concentrated at the center of mass) and the angular momentum of the system *about* its center of mass:

$$\mathbf{H}_O = \mathbf{X}_{\text{cm}} \times m \dot{\mathbf{X}}_{\text{cm}} + \sum_{i=1}^N \bar{\mathbf{X}}_i \times m_i \dot{\bar{\mathbf{X}}}_i.$$

Differentiating, we obtain

$$\dot{\mathbf{H}}_O = \dot{\mathbf{X}}_{\text{cm}} \times m\dot{\mathbf{X}}_{\text{cm}} + \mathbf{X}_{\text{cm}} \times m\ddot{\mathbf{X}}_{\text{cm}} + \sum_{i=1}^N \dot{\mathbf{X}}_i \times m_i \dot{\mathbf{X}}_i + \sum_{i=1}^N \bar{\mathbf{X}}_i \times m_i \ddot{\mathbf{X}}_i.$$

Because the cross-product of a vector with itself is zero, the first and third terms vanish. Thus,

$$\dot{\mathbf{H}}_O = \mathbf{X}_{\text{cm}} \times m\ddot{\mathbf{X}}_{\text{cm}} + \sum_{i=1}^N \bar{\mathbf{X}}_i \times m_i \ddot{\mathbf{X}}_i.$$

From (3), the first term is simply

$$\mathbf{X}_{\text{cm}} \times \mathbf{F}.$$

The second term is

$$\begin{aligned} \sum_{i=1}^N \bar{\mathbf{X}}_i \times m_i \ddot{\mathbf{X}}_i &= \sum_{i=1}^N \bar{\mathbf{X}}_i \times m_i (\ddot{\mathbf{X}}_i - \ddot{\mathbf{X}}_{\text{cm}}) \\ &= \left( \sum_{i=1}^N \bar{\mathbf{X}}_i \times m_i \ddot{\mathbf{X}}_i \right) - \left( \sum_{i=1}^N m_i \bar{\mathbf{X}}_i \right) \times \ddot{\mathbf{X}}_{\text{cm}} \\ &= \sum_{i=1}^N \bar{\mathbf{X}}_i \times m_i \ddot{\mathbf{X}}_i \end{aligned}$$

Applying Newton's second law of motion to each particle in the expression above, we find that

$$\dot{\mathbf{H}}_O = \mathbf{X}_{\text{cm}} \times \mathbf{F} + \underbrace{\sum_{i=1}^N \bar{\mathbf{X}}_i \times \left( \mathbf{F}_i + \sum_{j=1}^N \mathbf{F}_{ij} \right)}_{\mathbf{M}_{\text{cm}}}.$$

In words, the rate of change of total angular momentum about the origin of the inertial reference frame is equal to the sum of

1. the moment of all external forces about the origin of the inertial reference frame and
2. the moment of all (internal and external) forces *about the center of mass*.

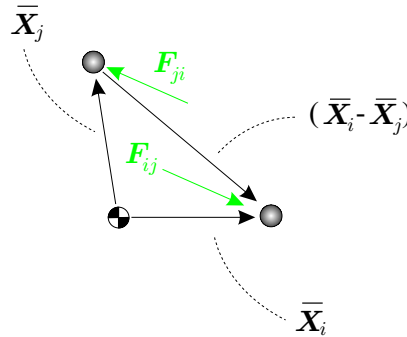


Figure 5: Forces of interaction between the  $i^{\text{th}}$  and  $j^{\text{th}}$  particles.

Recall the assumption that  $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$ . We may therefore rewrite the contributions to  $\mathbf{M}_{\text{cm}}$  due to the internal forces as follows

$$\begin{aligned} \sum_{i=1}^N \sum_{j=1}^N (\bar{\mathbf{X}}_i \times \mathbf{F}_{ij}) &= \sum_{i=1}^N \sum_{j=1}^i \{(\bar{\mathbf{X}}_i \times \mathbf{F}_{ij}) + (\bar{\mathbf{X}}_j \times \mathbf{F}_{ji})\} \\ &= \sum_{i=1}^N \sum_{j=1}^i \{(\bar{\mathbf{X}}_i \times \mathbf{F}_{ij}) + (\bar{\mathbf{X}}_j \times (-\mathbf{F}_{ij}))\} \\ &= \sum_{i=1}^N \sum_{j=1}^i \{(\bar{\mathbf{X}}_i - \bar{\mathbf{X}}_j) \times \mathbf{F}_{ij}\}. \end{aligned}$$

Assuming that the forces of interaction between particles act along the line joining the particles, these terms vanish leaving

$$\mathbf{M}_{\text{cm}} = \sum_{i=1}^N (\bar{\mathbf{X}}_i \times \mathbf{F}_i).$$

Summarizing,

$$\dot{\mathbf{H}}_{\text{O}} = \mathbf{X}_{\text{cm}} \times \mathbf{F} + \mathbf{M}_{\text{cm}} \quad (5)$$

where the first term is the moment of the resultant external force about the origin of the inertial frame and the second term is the resultant moment of all external forces about the center of mass.

Everything we have said thus far applies to a system of particles in unconstrained motion. The only assumption we have made is that the force of interaction between every two particles is equal and opposite and that the line of action of this force passes through the points. The conclusions are quite general.

In the next lecture, we will consider a collection of particles which are rigidly constrained with respect to one another. Starting from equation (5), we will then obtain a rotational version of equation (4) for a rigid body. Along the way, we will define an important geometric object known as the *inertia tensor*. Ultimately, we will obtain six differential equations that completely describe the dynamics of a rigid body moving under the influence of external forces and moments.