AOE 3134 Homework #5 Solutions

Problem 1. The dynamic equations for a freely rotating rigid body are

$$\dot{\boldsymbol{\omega}} = \boldsymbol{I}^{-1}\left((\boldsymbol{I}\boldsymbol{\omega}) \times \boldsymbol{\omega}\right).$$

Assume that the body-fixed reference frame is fixed in the principal axes of inertia so that $\boldsymbol{I} = \mathrm{diag}(I_x, I_y, I_z)$. Linearize the equations about the equilibrium $\boldsymbol{\omega}_{\mathrm{eq}} = [\omega_0,\ 0,\ 0]^T$. (Note that there will be no input matrix $\boldsymbol{B}$ for this problem.)

Solution. Writing out the equations explicitly, we have

$$\underbrace{\begin{pmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{pmatrix}}_{\dot{\boldsymbol{x}}} = \underbrace{\begin{pmatrix} \frac{I_y - I_z}{I_x}\omega_y\omega_z \\ \frac{I_z - I_x}{I_y}\omega_x\omega_z \\ \frac{I_x - I_y}{I_z}\omega_x\omega_y \end{pmatrix}}_{\boldsymbol{f}(\boldsymbol{x})}$$

Computing the jacobian of $\boldsymbol{f}(\boldsymbol{x})$, we find

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} = \begin{pmatrix} 0 & \frac{I_y - I_z}{I_x}\omega_z & \frac{I_y - I_z}{I_x}\omega_y \\ \frac{I_z - I_x}{I_y}\omega_z & 0 & \frac{I_z - I_x}{I_y}\omega_x \\ \frac{I_x - I_y}{I_z}\omega_y & \frac{I_x - I_y}{I_z}\omega_x & 0 \end{pmatrix}.$$

Evaluating at the nominal condition gives

$$\boldsymbol{A} = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{I_z - I_x}{I_y}\omega_0 \\ 0 & \frac{I_x - I_y}{I_z}\omega_0 & 0 \end{pmatrix}.$$

The linearized dynamics are therefore

$$\dot{\boldsymbol{\Delta x}} = \boldsymbol{A}\boldsymbol{\Delta x} \qquad \text{where} \qquad \boldsymbol{\Delta x} = \boldsymbol{x} - \boldsymbol{x}_0 = \begin{pmatrix} \omega_x - \omega_0 \\ \omega_y \\ \omega_z \end{pmatrix}.$$

To construct an actual state history $\boldsymbol{x}(t)$ in response to a given initial perturbation $\boldsymbol{\Delta x}(0)$, one would solve the linear, time-invariant equations for $\boldsymbol{\Delta x}(t)$ and add the result to the nominal value $\boldsymbol{x}_0 = \boldsymbol{\omega}_{\mathrm{eq}}$.

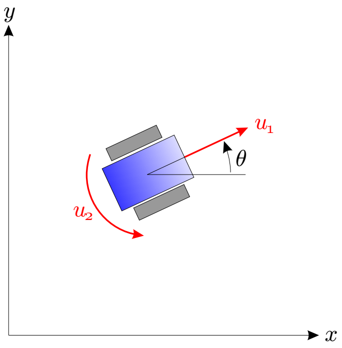

Figure 1: Kinematic model of a mobile robot.

Problem 2. A simple, kinematic model for a mobile robot is

$$\underbrace{\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix}}_{\dot{\boldsymbol{x}}} = \begin{pmatrix} \cos x_3 & 0 \\ \sin x_3 & 0 \\ 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}}_{\boldsymbol{u}}$$

where $x_1 = x$ and $x_2 = y$ denote the position of the robot and $x_3 = \theta$ denotes its orientation. The inputs are forward speed u_1 and turn rate u_2. Linearize these equations about the following two trajectories:

$$(a) \quad \boldsymbol{x}_{\mathrm{e}} = \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{u}_{\mathrm{e}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad (b) \quad \boldsymbol{x}_{\mathrm{e}} = \begin{pmatrix} \frac{1}{\omega} \sin \omega t \\ -\frac{1}{\omega} \cos \omega t \\ \omega t \end{pmatrix}, \quad \boldsymbol{u}_{\mathrm{e}} = \begin{pmatrix} 1 \\ \omega \end{pmatrix}$$

Solution. Writing out the equations explicitly, we have

$$\underbrace{\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix}}_{\dot{\boldsymbol{x}}} = \underbrace{\begin{pmatrix} u_1 \cos x_3 \\ u_1 \sin x_3 \\ u_2 \end{pmatrix}}_{\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u})}$$

Computing the jacobians of $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$ with respect to $\boldsymbol{x}$ and $\boldsymbol{u}$, we find

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} = \begin{pmatrix} 0 & 0 & -u_1 \sin x_3 \\ 0 & 0 & u_1 \cos x_3 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}} = \begin{pmatrix} \cos x_3 & 0 \\ \sin x_3 & 0 \\ 0 & 1 \end{pmatrix}.$$

Evaluating along the first solution gives

$$\boldsymbol{A} = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{B} = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\right|_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, we obtain linear, time-invariant perturbation equations

$$\dot{\boldsymbol{\Delta x}} = \boldsymbol{A}\boldsymbol{\Delta x} + \boldsymbol{B}\boldsymbol{\Delta u}.$$

Evaluating the jacobians along the second solution gives

$$\boldsymbol{A} = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_0 = \begin{pmatrix} 0 & 0 & -\sin \omega t \\ 0 & 0 & \cos \omega t \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{B} = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\right|_0 = \begin{pmatrix} \cos \omega t & 0 \\ \sin \omega t & 0 \\ 0 & 1 \end{pmatrix}.$$

In this case, we obtain linear, time-*varying* perturbation equations

$$\dot{\boldsymbol{\Delta x}} = \boldsymbol{A}(t)\boldsymbol{\Delta x} + \boldsymbol{B}(t)\boldsymbol{\Delta u}.$$

Problem 3. Using the Taylor series approach, verify the six linearized kinematic equations given in Lecture 14.

Solution. Not provided. (You were simply asked to verify a given set of equations using an alternate approach.)