# AOE 3134 Homework #4

Assigned: March 13, 2007
Due: March 20, 2007 (Place your homework in the box outside my office by 5 PM.)

**Problem 1.** *Part 1)* Consider the following system of linear ordinary differential equations:

$$\ddot{x} + a\dot{y} + b\dot{x} + cx = 0$$
$$\ddot{y} + d\dot{x} + e\dot{y} + fy = 0$$

where $a, b, c, d, e$, and $f$ are constant coefficients. Find conditions on these coefficients for static stability. (Consider "perturbed" initial states of the form $(x(0), y(0), \dot{x}(0), \dot{y}(0)) = (x_0, y_0, 0, 0)$ as in Lecture 10.)

*Part 2)* Make the change of variables $x_1 = x$, $x_2 = y$, $x_3 = \dot{x}$, $x_4 = \dot{y}$ and introduce the *state vector* $\boldsymbol{x} = [x_1,\ x_2,\ x_3,\ x_4]^T$. With this change of variables, one obtains a system of first order ODEs:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$$

where $\boldsymbol{A}$ is a $4 \times 4$ matrix of real numbers. For time-invariant systems such as this, the eigenvalues of the *state matrix* $\boldsymbol{A}$ determine dynamic stability. Because the eigenvalues appear as the arguments of exponentials in the system's transient response, *a necessary and sufficient condition for asymptotic stability is that every eigenvalue must have strictly negative real part.*

i) Give the explicit expression for $\boldsymbol{A}$ in terms of the parameters $a$ through $f$.

ii) Assess dynamic (asymptotic) stability by computing the eigenvalues of $\boldsymbol{A}$ in the case that

$$a = d = 0, \quad b = 0.2, \quad c = 1, \quad e = 0.4, \quad \text{and} \quad f = 4.$$

iii) Repeat (ii) in the case where $a = d = 2$, with the remaining parameters as given.

**Problem 2.** Consider a rigid body with inertia matrix $\boldsymbol{I} = \text{diag}(I_x, I_y, I_z)$ where $I_x$, $I_y$, and $I_z$ are the positive-valued *principal moments of inertia.* Assuming small perturbations from the equilibrium $\boldsymbol{\omega}_{\text{eq}} = [\omega_0,\ 0,\ 0]^T$, the approximate (linearized) dynamic equations for free rotational motion are

$$\frac{d}{dt}\boldsymbol{\Delta\omega} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \omega_0(I_z - I_x)/I_y \\ 0 & \omega_0(I_x - I_y)/I_z & 0 \end{pmatrix} \boldsymbol{\Delta\omega}.$$

The true angular rate $\boldsymbol{\omega}$ is the sum of the nominal term $\boldsymbol{\omega}_{\text{eq}}$ and the perturbation term $\boldsymbol{\Delta\omega}$. Noting that the perturbation dynamics are linear, time-invariant, use spectral analysis as in Problem 1 to investigate stability. Consider three cases: (i) $I_x < I_y < I_z$, (ii) $I_y < I_x < I_z$, and (iii) $I_y < I_z < I_x$. *Note:* For nonlinear systems, stability (instability) of the linearized dynamics implies stability (instability) of the nonlinear dynamics *except* in situations where (a) none of the eigenvalues lie in the open right half plane *and* (b) some eigenvalues lie on the imaginary axis. If such a case arises, write "inconclusive."

**Problem 3.** An aircraft in longitudinal flight (not necessarily equilibrium flight) is equipped with a laser range finder that determines range $R$ and elevation angle $\eta$ to a surface object, as well as $\dot{R}$ and $\dot{\eta}$. Assume that the sensor is located at the origin of the body reference frame and that $\eta$ is measured from the body $\boldsymbol{z}_{\text{B}}$ axis and in the same sense as the body's pitch angle $\theta$. Suppose that the aircraft overflies a marine surface vehicle that is moving in the same direction and in the same vertical plane. Develop expressions for the surface vehicle's inertial horizontal position $x_{\text{s}}$ and horizontal velocity $\dot{x}_{\text{s}}$ in terms of the aircraft's state and the measurements. That is, find expressions for $x_{\text{s}}$ and $\dot{x}_{\text{s}}$ in terms of the state variables $x$, $z$, $\theta$, $u$, $w$, and $q$ and the measurements $R$ and $\eta$ and their time derivatives.