AOE 3134 Homework #4 Solutions

Problem 1. *Part 1)* Consider the following system of linear ordinary differential equations:

$$\ddot{x} + a\dot{y} + b\dot{x} + cx = 0$$
$$\ddot{y} + d\dot{x} + e\dot{y} + fy = 0$$

where a, b, c, d, e, and f are constant coefficients. Find conditions on these coefficients for static stability. (Consider "perturbed" initial states of the form $(x(0), y(0), \dot{x}(0), \dot{y}(0)) = (x_0, y_0, 0, 0)$ as in Lecture 10.)

Part 2) Make the change of variables $x_1 = x$, $x_2 = y$, $x_3 = \dot{x}$, $x_4 = \dot{y}$ and introduce the *state vector* $\boldsymbol{x} = [x_1,\ x_2,\ x_3,\ x_4]^T$. With this change of variables, one obtains a system of first order ODEs:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$$

where $\boldsymbol{A}$ is a 4×4 matrix of real numbers. For time-invariant systems such as this, the eigenvalues of the *state matrix* $\boldsymbol{A}$ determine dynamic stability. Because the eigenvalues appear as the arguments of exponentials in the system's transient response, *a necessary and sufficient condition for asymptotic stability is that every eigenvalue must have strictly negative real part.*

i) Give the explicit expression for $\boldsymbol{A}$ in terms of the parameters a through f.

ii) Assess dynamic (asymptotic) stability by computing the eigenvalues of $\boldsymbol{A}$ in the case that

$$a = d = 0, \quad b = 0.2, \quad c = 1, \quad e = 0.4, \quad \text{and} \quad f = 4.$$

iii) Repeat (ii) in the case where $a = d = 2$, with the remaining parameters as given.

Solution. Consider initial perturbations of the form $(x(0), y(0), \dot{x}(0), \dot{y}(0)) = (x_0, y_0, 0, 0)$ as suggested, we have

$$\ddot{x}(0) = -cx(0) \qquad \text{and} \qquad \ddot{y}(0) = -fy(0).$$

Thus, the conditions for static stability are $c > 0$ and $f > 0$ so that

$$\ddot{x}(0) = -cx(0) \begin{cases} < 0 & x_0 > 0 \\ > 0 & x_0 < 0 \end{cases} \ddot{y}(0) = -fy(0) \begin{cases} < 0 & y_0 > 0 \\ > 0 & y_0 < 0 \end{cases}$$

Making the suggested change of variables, we obtain the system

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} \qquad \text{where} \qquad \boldsymbol{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -c & 0 & -b & -a \\ 0 & -f & -d & -e \end{pmatrix}$$

The eigenvalues corresponding to the parameter values in (ii) are

$$\lambda_{1,2} = -0.1 \pm j1.0 \qquad \text{where} \qquad \lambda_{3,4} = -0.2 \pm j2.0$$

(These values, computed using Matlab, have been rounded to one decimal place.) Because every eigenvalue has strictly negative real part, the system is dynamically stable for these parameter values.

For the parameter values in (iii), the eigenvalues are

$$\lambda_{1,2} = 0.7 \pm j1.2 \qquad \text{where} \qquad \lambda_{3,4} = -1.0 \pm j1.0$$

The system is *unstable* for these parameter values.

Problem 2. Consider a rigid body with inertia matrix $\boldsymbol{I} = \text{diag}(I_x, I_y, I_z)$ where I_x, I_y, and I_z are the positive-valued *principal moments of inertia*. Assuming small perturbations from the equilibrium $\boldsymbol{\omega}_{\text{eq}} = [\omega_0,\ 0,\ 0]^T$, the approximate (linearized) dynamic equations for free rotational motion are

$$\frac{d}{dt}\boldsymbol{\Delta\omega} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \omega_0(I_z - I_x)/I_y \\ 0 & \omega_0(I_x - I_y)/I_z & 0 \end{pmatrix} \boldsymbol{\Delta\omega}.$$

The true angular rate $\boldsymbol{\omega}$ is the sum of the nominal term $\boldsymbol{\omega}_{\text{eq}}$ and the perturbation term $\boldsymbol{\Delta\omega}$. Noting that the perturbation dynamics are linear, time-invariant, use spectral analysis as in Problem 1 to investigate stability. Consider three cases: (i) $I_x < I_y < I_z$, (ii) $I_y < I_x < I_z$, and (iii) $I_y < I_z < I_x$. *Note:* For nonlinear systems, stability (instability) of the linearized dynamics implies stability (instability) of the nonlinear dynamics *except* in situations where (a) none of the eigenvalues lie in the open right half plane *and* (b) some eigenvalues lie on the imaginary axis. If such a case arises, write "inconclusive."

Solution. The characteristic polynomial is

$$\det\left(\lambda\mathbb{I} - \boldsymbol{A}\right) = \lambda \left(\lambda^2 - \underbrace{\frac{\omega_0^2}{I_y I_x}\left((I_z - I_x)(I_x - I_y)\right)}_{\alpha} \right).$$

In case (i), corresponding to steady rotation about the *minor* principal axis of inertia, $\alpha < 0$ and all three characteristic values lie on the imaginary axis; nonlinear stability is *inconclusive*. The same is true in case (iii), which corresponds to steady rotation about the *major* principal axis of inertia. A more sophisticated, nonlinear stability analysis verifies that the equilibrium is indeed stable, but not asymptotically stable; a small perturbation from the equilibrium will result in small oscillations of $\boldsymbol{\omega}$ about its equilibrium value. In case (ii), corresponding to steady rotation about the *intermediate* principal axis of inertia, $\alpha > 0$ resulting in a real conjugate pair of eigenvalues. Since one eigenvalue is in the right half plane, this equilibrium is unstable. A small perturbation from the equilibrium value, in this case, gives rise to a trajectory $\boldsymbol{\omega}(t)$ which diverges from the equilibrium value; the corresponding motion looks very little like steady rotation about the intermediate axis.

Problem 3. An aircraft in longitudinal flight (not necessarily equilibrium flight) is equipped with a laser range finder that determines range R and elevation angle η to a surface object, as well as $\dot{R}$ and $\dot{\eta}$. Assume that the sensor is located at the origin of the body reference frame and that η is measured from the body $\boldsymbol{z}_{\text{B}}$ axis and in the same sense as the body's pitch angle θ. Suppose that the aircraft overflies a marine surface vehicle that is moving in the same direction and in the same vertical plane. Develop expressions for the surface vehicle's inertial horizontal position x_{s} and horizontal velocity $\dot{x}_{\text{s}}$ in terms of the aircraft's state and the measurements. That is, find expressions for x_{s} and $\dot{x}_{\text{s}}$ in terms of the state variables x, z, θ, u, w, and q and the measurements R and η and their time derivatives.

Solution. Let $\boldsymbol{x}_{\text{s}}$ be the position of the surface vehicle relative to the origin of the inertial frame (and expressed in the inertial frame). Then

$$\boldsymbol{x}_{\text{s}} = \boldsymbol{x} + \boldsymbol{R}_{\text{IB}}(\theta)\boldsymbol{R}_{\text{BS}}(\eta) \begin{pmatrix} 0 \\ 0 \\ R \end{pmatrix}_{\text{S}}$$

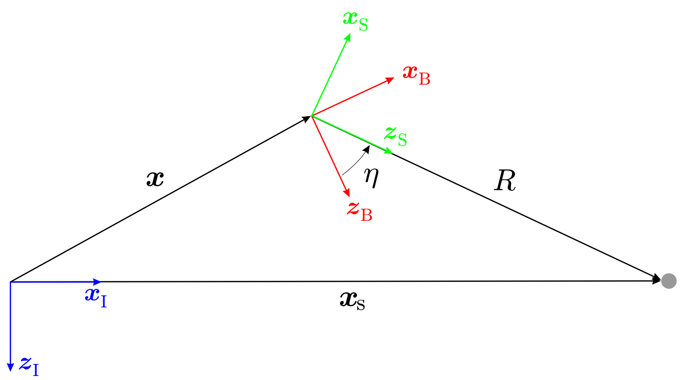

Figure 1: Reference frames for Problem 3.

where the subscript "S" denotes the "sensor frame" as depicted in Figure 1. The matrices above are

$$\boldsymbol{R}_\mathrm{IB}(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \quad \text{and} \quad \boldsymbol{R}_\mathrm{BS}(\eta) = \begin{pmatrix} \cos\eta & 0 & \sin\eta \\ 0 & 1 & 0 \\ -\sin\eta & 0 & \cos\eta \end{pmatrix}.$$

It is easy to verify, using common trigonometric identities, that

$$\boldsymbol{R}_\mathrm{IS}(\theta,\eta) = \boldsymbol{R}_\mathrm{IB}(\theta)\boldsymbol{R}_\mathrm{BS}(\eta) = \begin{pmatrix} \cos(\theta+\eta) & 0 & \sin(\theta+\eta) \\ 0 & 1 & 0 \\ -\sin(\theta+\eta) & 0 & \cos(\theta+\eta) \end{pmatrix}.$$

It follows that

$$x_\mathrm{s} = \boldsymbol{x}_\mathrm{s} \cdot \boldsymbol{x}_\mathrm{I} = \boldsymbol{x}_\mathrm{s} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_\mathrm{I} = x + R\sin(\theta+\eta).$$

Since $\boldsymbol{x}_\mathrm{s}$ is already expressed in the inertial frame, the brute force approach to computing the surface vessel's velocity is simply to differentiate:

$$\begin{aligned} \dot{\boldsymbol{x}}_\mathrm{s} &= \dot{\boldsymbol{x}} + \boldsymbol{R}_\mathrm{IS}(\theta,\eta)\begin{pmatrix} 0 \\ 0 \\ \dot{R} \end{pmatrix}_\mathrm{S} + \dot{\boldsymbol{R}}_\mathrm{IS}(\theta,\eta)\begin{pmatrix} 0 \\ 0 \\ R \end{pmatrix}_\mathrm{S} \\ &= \boldsymbol{R}_{IB}(\theta)\boldsymbol{v} + \boldsymbol{R}_\mathrm{IS}(\theta,\eta)\begin{pmatrix} 0 \\ 0 \\ \dot{R} \end{pmatrix}_\mathrm{S} + \boldsymbol{R}_\mathrm{IS}(\theta,\eta)\begin{pmatrix} 0 & 0 & (\dot{\theta}+\dot{\eta}) \\ 0 & 0 & 0 \\ -(\dot{\theta}+\dot{\eta}) & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ R \end{pmatrix}_\mathrm{S} \end{aligned} \tag{1}$$

Performing the calculations and taking the first (horizontal) component gives

$$\dot{x}_\mathrm{s} = (u\cos\theta + w\sin\theta) + \dot{R}\sin(\theta+\eta) + R\cos(\theta+\eta)(q+\dot{\eta}).$$

To try and connect this to our discussion of derivatives in rotating frames, notice that premultiplying (1) by $\boldsymbol{R}_{BI}(\theta) = \boldsymbol{R}_{IB}(\theta)^T$ gives

$$\boldsymbol{R}_{BI}(\theta)\dot{\boldsymbol{x}}_\mathrm{s} = \boldsymbol{v} + \boldsymbol{R}_\mathrm{BS}(\eta)\begin{pmatrix} 0 \\ 0 \\ \dot{R} \end{pmatrix}_\mathrm{S} + \boldsymbol{R}_\mathrm{BS}(\eta)\left[\begin{pmatrix} 0 \\ (q+\dot{\eta}) \\ 0 \end{pmatrix}_\mathrm{S} \times \begin{pmatrix} 0 \\ 0 \\ R \end{pmatrix}_\mathrm{S}\right]$$

The first term on the right hand side is the velocity of the body relative to the inertial frame (but expressed in the body frame). The second term is the velocity of the surface vessel relative to the body frame. The third term is the component of velocity due to the sensor's rotation relative to the inertial frame at rate $\dot{\theta}+\dot{\eta}$ about the (inertial, body, or sensor) $\boldsymbol{y}$-axis.