AOE 3134
Stability and Control
Final Exam Solutions

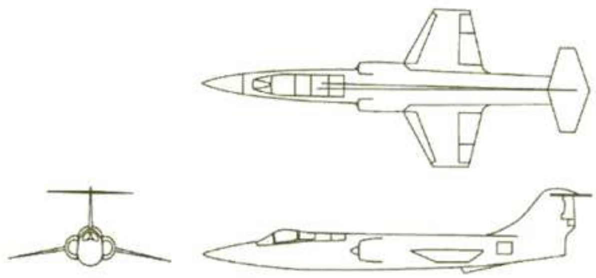

Figure 1: F-104A.

Problem 1. (30 points) Table 1 gives the longitudinal stability coefficients for an F-104A fighter flying horizontally at Mach 0.26 at sea level conditions.[1] All angles are measured in radians.

Table 1: F-104A Longitudinal Stability Coefficients.

	0 (nominal)	α	$\dot{\alpha}$	q	Ma	δe
$C_{D_{(\cdot)}}$	0.263	0.45	0	0	0	0
$C_{L_{(\cdot)}}$	0.833	3.44	0	0	0	0.68
$C_{m_{(\cdot)}}$	0	−0.64	−1.6	−5.8	0	−1.46

Other relevant parameters include:

$$W = 16,300 \text{ lbs} \quad I_x = 3,550 \text{ slug ft}^2 \quad I_y = 58,600 \text{ slug ft}^2 \quad I_z = 59,700 \text{ slug ft}^2 \quad I_{xz} = 0 \text{ slug ft}^2$$

$$S = 200 \text{ ft}^2 \quad b = 22 \text{ ft} \quad \text{and} \quad \bar{c} = 10 \text{ ft}.$$

The stability coefficients given above are computed in the stability axes. The variables α and δe are thus measured from their equilibrium values; $\alpha = 0$ and $\delta e = 0$ corresponds to the given equilibrium flight condition. (Also note that C_{m_0} is zero in stability axes.)

A. (10 points) When the airplane returns from a sortie, it is 1300 pounds lighter. Compute the new equilibrium value of α and δe. Assume the same speed and altitude.

B. (20 points) Compute approximate values for the natural frequency and damping ratio of the phugoid and short period modes.

Solution.
A. (10 points) The airspeed is

$$u_0 = \text{Ma} \cdot a = 287 \text{ ft/s}.$$

The new weight $W = 15,000$ lbs gives

$$C_W = \frac{15,000}{\frac{1}{2}\rho u_0^2 S} = 0.761.$$

[1] For the standard atmosphere at sea level, density is $\rho \approx 2.38 \times 10^{-3}$ slugs/ft^3 and the speed of sound is $a \approx 1120$ ft/s.

The revised trim conditions are

$$\begin{pmatrix} \alpha \\ \delta e \end{pmatrix} = \begin{pmatrix} C_{L_\alpha} & C_{L_{\delta e}} \\ C_{m_\alpha} & C_{m_{\delta e}} \end{pmatrix}^{-1} \begin{pmatrix} C_W - C_{L_0} \\ 0 \end{pmatrix} = \begin{pmatrix} -0.021 \\ 0.009 \end{pmatrix} \text{ rad} = \begin{pmatrix} -1.2^\circ \\ 0.5^\circ \end{pmatrix}.$$

B. (20 points) The short period natural frequency and damping ratio approximation is

$$\omega_{n_{\rm sp}} \approx \sqrt{\frac{1}{mI_y} Z_w M_q - \frac{u_0}{I_y} M_w} \quad \text{and} \quad \zeta_{\rm sp} \approx -\frac{1}{2\omega_{n_{\rm sp}}} \left[\frac{1}{m} Z_w + \frac{1}{I_y} \left(M_q + u_0 M_{\dot{w}} \right) \right].$$

The phugoid natural frequency and damping ratio approximation is

$$\omega_{n_{\rm p}} \approx \sqrt{-\frac{Z_u g}{m u_0}} \quad \text{and} \quad \zeta_{\rm p} \approx -\frac{X_u}{2m\omega_{n_{\rm P}}}.$$

We must therefore compute

$$\begin{aligned}
m &= \frac{W}{g} && = 506 \text{ slugs with } W = 16,300 \text{ lbs } \textit{or} \\
& && = 466 \text{ slugs with } W = 15,000 \text{ lbs} \\
Z_u &= -\frac{1}{2}\rho u_0 S \left(2C_{L_0} + C_{L_u} \right) && = -113 \text{ lbs/(ft/s)} \\
X_u &= \frac{1}{2}\rho u_0 S \left[2\left(-C_{D_0} + C_{T_0} \right) + \left(-C_{D_u} + C_{T_u} \right) \right] && = -36 \text{ lbs/(ft/s)} \\
Z_w &= -\frac{1}{2}\rho u_0 S \left(C_{D_0} + C_{L_\alpha} \right) && = -254 \text{ lbs/(ft/s)} \\
M_w &= \frac{1}{2}\rho u_0 S \bar{c} C_{m_\alpha} && = -438 \text{ (ft} \cdot \text{lbs)/(ft/s)} \\
M_q &= \frac{1}{4}\rho u_0 S \bar{c}^2 C_{m_q} && = -1990 \text{ (ft} \cdot \text{lbs)/(rad/s)} \\
M_{\dot{w}} &= \frac{1}{4}\rho S \bar{c}^2 C_{m_{\dot{\alpha}}} && = -19 \text{ (ft} \cdot \text{lbs)/(ft/s}^2)
\end{aligned}$$

Here, we have assumed that $C_{D_u} = C_{L_u} = C_{m_u}$ depend only on compressibility effects, and consequently are zero. Also, since the F-104 is a jet, we have assumed that it produces constant thrust so that $C_{T_u} = -2C_{T_0}$.

With $W = 16,300$, the resulting approximations are

$$\omega_{n_{\rm sp}} \approx 1.52 \text{ rad/s} \quad \text{and} \quad \zeta_{\rm sp} \approx 0.31$$

and

$$\omega_{n_{\rm p}} \approx 0.16 \text{ rad/s} \quad \text{and} \quad \zeta_{\rm p} \approx 0.23.$$

Alternatively, if you used $W = 15,000$, the resulting approximations are

$$\omega_{n_{\rm sp}} \approx 1.53 \text{ rad/s} \quad \text{and} \quad \zeta_{\rm sp} \approx 0.32$$

and

$$\omega_{n_{\rm p}} \approx 0.16 \text{ rad/s} \quad \text{and} \quad \zeta_{\rm p} \approx 0.23.$$

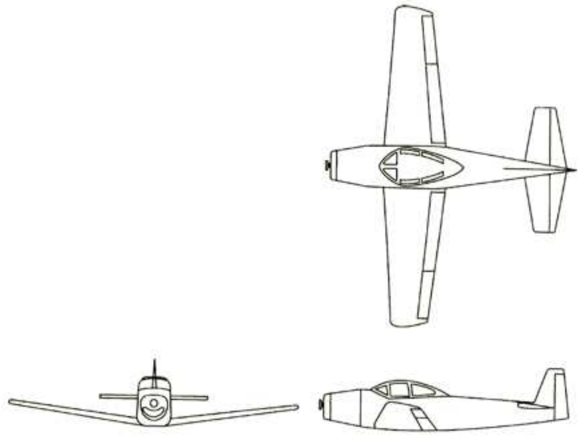

Figure 2: Navion.

Problem 2. (30 points) Table 2 gives the longitudinal stability coefficients for a Navion flying horizontally at Mach 0.16 at sea level conditions.[2] All angles are measured in radians.

Table 2: Navion Lateral-Directional Stability Coefficients.

	β	p	r	δa	δr
$C_{Y_{(\cdot)}}$	-0.564	0	0	0	0.157
$C_{l_{(\cdot)}}$	-0.074	-0.410	0.107	-0.134	0.107
$C_{n_{(\cdot)}}$	0.071	-0.0575	-0.125	-0.0035	-0.072

Other relevant parameters include:

$$W = 2750 \text{ lbs} \quad I_x = 1050 \text{ slug ft}^2 \quad I_y = 3000 \text{ slug ft}^2 \quad I_z = 3500 \text{ slug ft}^2 \quad I_{xz} = 0 \text{ slug ft}^2$$

$$S = 184 \text{ ft}^2 \quad b = 33 \text{ ft} \quad \text{and} \quad \bar{c} = 6 \text{ ft}.$$

A. (10 points) Compute the control deflections δa and δr required, and the corresponding roll angle ϕ, for this airplane to maintain heading given a 20 ft/s cross-wind coming from the right. (Assume that the total airspeed is the nominal speed.)

B. (20 points) Compute approximate values for the natural frequency and damping ratio of the Dutch roll mode and for the characteristic values corresponding to the spiral and rolling convergence modes.

Solution.
A. (10 points) For sea level flight at $M = 0.158$, we have $V = 179$ ft/s and the weight coefficient is therefore

$$C_W = \frac{2750}{\left(\frac{1}{2}(2.38E-3)(179)^2\right)(184)} = 0.391.$$

For a 20 ft/s cross-wind,

$$\beta = \arcsin\left(\frac{20}{179}\right) = 0.112 \text{ rad} \approx 6.4^\circ.$$

[2]For the standard atmosphere at sea level, density is $\rho \approx 2.38 \times 10^{-3}$ slugs/ft^3 and the speed of sound is $a \approx 1120$ ft/s.

Solving

$$\begin{pmatrix} 0 & 0.157 & 0.391 \\ -0.134 & 0.107 & 0 \\ -0.0035 & -0.072 & 0 \end{pmatrix} \begin{pmatrix} \delta a \\ \delta r \\ \phi \end{pmatrix} = - \begin{pmatrix} -0.564 \\ -0.074 \\ 0.071 \end{pmatrix} (0.112)$$

gives

$$\begin{pmatrix} \delta a \\ \delta r \\ \phi \end{pmatrix} = \begin{pmatrix} 0.025 \\ 0.109 \\ 0.118 \end{pmatrix} \text{rad} \approx \begin{pmatrix} 1.5^\circ \\ 6.2^\circ \\ 6.7^\circ \end{pmatrix}$$

B. (20 points) First, recall the following approximate formulas. For the spiral mode,

$$\lambda_{\rm s} \approx \frac{(\mathcal{L}_r \mathcal{N}_v - \mathcal{L}_v \mathcal{N}_r)}{-I_z \mathcal{L}_v + \frac{u_0}{g} (\mathcal{L}_v \mathcal{N}_p - \mathcal{L}_p \mathcal{N}_v)}.$$

For the rolling convergence mode,

$$\lambda_{\rm r} \approx \frac{1}{I_x} \mathcal{L}_p.$$

For the Dutch roll mode,

$$\omega_{n_{\rm dr}} \approx \sqrt{\frac{1}{m I_z} (Y_v \mathcal{N}_r - \mathcal{N}_v Y_r) + \frac{u_0}{I_z} \mathcal{N}_v} \qquad \text{and} \qquad \zeta_{\rm dr} \approx -\frac{1}{2\omega_{n_{\rm dr}}} \left(\frac{1}{m} Y_v + \frac{1}{I_z} \mathcal{N}_r \right).$$

Since $I_{xz} = 0$, it follows that $\mathcal{L}_{(\cdot)} = L_{(\cdot)}$ and $\mathcal{N}_{(\cdot)} = N_{(\cdot)}$. We therefore compute

$$\begin{aligned}
m &= \frac{W}{g} && = 85 \text{ slugs} \\
Y_v &= \frac{1}{2} \rho u_0 S C_{Y_\beta} && = -22 \text{ lbs/(ft/s)} \\
L_v &= \frac{1}{2} \rho u_0 b S C_{l_\beta} && = -96 \text{ (ft} \cdot \text{lbs)/(ft/s)} \\
N_v &= \frac{1}{2} \rho u_0 b S C_{n_\beta} && = 92 \text{ (ft} \cdot \text{lbs)/(ft/s)} \\
Y_p &= \frac{1}{4} \rho u_0 b S C_{Y_p} && = 0 \text{ lbs/(rad/s)} \\
L_p &= \frac{1}{4} \rho u_0 b^2 S C_{l_p} && = -8760 \text{ (ft} \cdot \text{lbs)/(rad/s)} \\
N_p &= \frac{1}{4} \rho u_0 b^2 S C_{n_p} && = -1230 \text{ (ft} \cdot \text{lbs)/(rad/s)} \\
Y_r &= \frac{1}{4} \rho u_0 b S C_{Y_r} && = 0 \text{ lbs/(rad/s)} \\
L_r &= \frac{1}{4} \rho u_0 b^2 S C_{l_r} && = 2290 \text{ (ft} \cdot \text{lbs)/(ft/s)} \\
N_r &= \frac{1}{4} \rho u_0 b^2 S C_{n_r} && = -2670 \text{ (ft} \cdot \text{lbs)/(ft/s)}
\end{aligned}$$

Substituting into the expressions above gives

$$\lambda_{\rm s} \approx -0.007$$

For the rolling convergence mode,

$$\lambda_{\rm r} \approx -27.8$$

For the Dutch roll mode,

$$\omega_{n_{\rm dr}} \approx 1.21 \text{ rad/s} \qquad \text{and} \qquad \zeta_{\rm dr} \approx 0.20$$

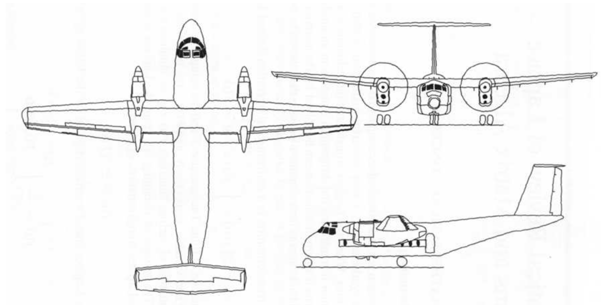

Figure 3: Short takeoff and landing aircraft.

Problem 3. (30 points) Table 3 gives the longitudinal stability coefficients for a short takeoff and landing aircraft flying horizontally at Mach 0.14 at sea level conditions.[3] All angles are measured in radians.

Table 3: Navion Lateral-Directional Stability Coefficients.

	β	p	r	δa	δr
$C_{Y_{(\cdot)}}$	-0.564	0	0	0	0.157
$C_{l_{(\cdot)}}$	-0.074	-0.410	0.107	-0.134	0.107
$C_{n_{(\cdot)}}$	0.071	-0.0575	-0.125	-0.0035	-0.072

Other relevant parameters include:

$$W = 40,000 \text{ lbs} \quad I_x = 270 \cdot 10^3 \text{ slug ft}^2 \quad I_y = 220 \cdot 10^3 \text{ slug ft}^2 \quad I_z = 450 \cdot 10^3 \text{ slug ft}^2 \quad I_{xz} = 0 \text{ slug ft}^2$$

$$S = 950 \text{ ft}^2 \quad b = 100 \text{ ft} \quad \text{and} \quad \bar{c} = 10 \text{ ft}.$$

A. (15 points) The roll rate is well-approximated by the first order equation

$$I_x \dot{p} = \mathcal{L}_p p + \mathcal{L}_{\delta a} \delta a \quad \text{where} \quad \mathcal{L}_{(\cdot)} = \frac{I_x}{I_\mathrm{x} I_\mathrm{z} - I_\mathrm{xz}^2} \left(I_\mathrm{z} L_{(\cdot)} + I_\mathrm{xz} N_{(\cdot)} \right).$$

Use this approximation to compute the roll rate response $p(t)$ to an impulse aileron input

$$\delta a(t) = 0.1 \; \delta(t) \text{ rad}.$$

with $p(0) = 0$. You may find it easier to solve the for the step response symbolically and then substitute values.

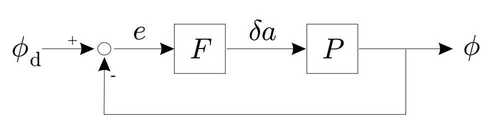

Figure 4: The closed-loop control system.

B. (15 points) Although an aircraft has no intrinsic roll stiffness, this effect can be synthesized by feeding roll angle measurements back to the ailerons. Using the approximate roll dynamics derived above, one finds that the transfer function from aileron deflection to roll angle is

$$P(s) = \frac{\phi(s)}{\delta a(s)} = \frac{\mathcal{L}_{\delta a}}{s \left(I_x s - \mathcal{L}_p \right)}.$$

[3] For the standard atmosphere at sea level, density is $\rho \approx 2.38 \times 10^{-3}$ slugs/ft^3 and the speed of sound is $a \approx 1120$ ft/s.

(Note that the pole at zero in the transfer function $P(s)$ indicates that the roll angle is only neutrally stable.) Consider the simple closed-loop control system shown in Figure 4 where the compensator $F(s)$ provides proportional-integral feedback:

$$F(s) = \frac{\delta a(s)}{e(s)} = k_p\left(1 + \frac{1}{s}\right)$$

where $e(s) = \phi_{\mathrm{d}}(s) - \phi(s)$ is the error in the commanded roll angle and k_p is a proportional gain.

Compute the (approximate) closed-loop transfer function

$$H(s) = \frac{\phi(s)}{\phi_{\mathrm{d}}(s)} = \frac{F(s)P(s)}{1 + F(s)P(s)}$$

explicitly (as a numerator polynomial divided by a denominator polynomial). Then use the Routh-Hurwitz method to obtain necessary and sufficient conditions on the proportional gain k_p for closed-loop stability.

Solution.

A. (15 points) The equation is

$$\dot{p} = ap + bu$$

where

$$a = \frac{\mathcal{L}_p}{I_x} \quad \text{and} \quad b = \frac{\mathcal{L}_{\delta a}}{I_x}.$$

The general solution can be found quite easily using, for example, the Laplace transform approach. Define $P(s) = \mathcal{L}\{p(t)\}$ and $U(s) = \mathcal{L}\{u(t)\}$. Then

$$sP(s) - p(0) = aP(s) + bU(s)$$

or

$$P(s) = \frac{b}{s-a}U(s).$$

The Laplace transform of the impulse function is $\mathcal{L}\{\delta(t)\} = 1$ so we have

$$\begin{aligned} p(t) = \mathcal{L}^{-1}\left\{0.1\,\frac{b}{s-a}\right\} &= 0.1\,be^{at} \\ &= 0.1\,\frac{\mathcal{L}_{\delta a}}{I_x}\exp\left(\frac{\mathcal{L}_p}{I_x}t\right). \end{aligned}$$

Note that the problem statement contained an incorrect definition for $\mathcal{L}_{(\cdot)}$. (The correct definition was given in the supplement.) For the correct definition, one should find that $\mathcal{L}_{(\cdot)} = L_{(\cdot)}$ when $I_{xz} = 0$. In any case, the solution given above is correct and warrants full credit regardless of the numerical values computed for $\mathcal{L}_p$ and $\mathcal{L}_{\delta a}$.

B. (15 points) The closed-loop transfer function is

$$\begin{aligned} H(s) = \frac{F(s)P(s)}{1+F(s)P(s)} &= \frac{\left(k_p\left(1+\frac{1}{s}\right)\right)\left(\frac{\mathcal{L}_{\delta a}}{s(I_x s - \mathcal{L}_p)}\right)}{1 + \left(k_p\left(1+\frac{1}{s}\right)\right)\left(\frac{\mathcal{L}_{\delta a}}{s(I_x s - \mathcal{L}_p)}\right)} \\ &= \frac{k_p\mathcal{L}_{\delta a}(s+1)}{s^2\left(I_x s - \mathcal{L}_p\right) + k_p\mathcal{L}_{\delta a}(s+1)} \\ &= \frac{k_p\mathcal{L}_{\delta a}(s+1)}{I_x s^3 - \mathcal{L}_p s^2 + k_p\mathcal{L}_{\delta a}s + k_p\mathcal{L}_{\delta a}} \end{aligned}$$

The closed-loop characteristic values are the roots of the denominator polynomial. Since $I_x > 0$, necessary conditions for stability are that

$$-\mathcal{L}_p > 0 \qquad \text{and} \qquad k_p \mathcal{L}_{\delta a} > 0.$$

The first condition is automatically satisfied. Since $\mathcal{L}_{\delta a} < 0$, this implies that k_p should be chosen negative. The complete set of (necessary and sufficient) conditions are obtained by performing a Routh-Hurwitz analysis. The Routh array is

$$\begin{array}{lcc} s^3 & I_x & k_p \mathcal{L}_{\delta a} \\ s^2 & -\mathcal{L}_p & k_p \mathcal{L}_{\delta a} \\ s^1 & \frac{1}{\mathcal{L}_p} \left(I_x + \mathcal{L}_p \right) k_p \mathcal{L}_{\delta a} & \\ s^0 & k_p \mathcal{L}_{\delta a} & \end{array}$$

Since we have already required that $k_p \mathcal{L}_{\delta a} > 0$, the only additional condition is that

$$I_x + \mathcal{L}_p < 0.$$

Thus, we must have sufficiently large roll damping or a sufficiently small roll inertia, either of which corresponds to a better damped (i.e., faster) rolling convergence mode.

Part 2 Instructions: Circle the letter corresponding to the best answer for ***ten*** of the following fifteen questions. ***Clearly mark the questions you want to have graded by circling the number next to the question.*** Each question is worth four points. (Don't forget to turn these answers in with your solutions to Part 1!)

1. Suppose that, at a given instant, a certain airplane's pitch angle θ passes through $0°$ and its roll angle ϕ passes through $90°$. Then, at this same instant,

   (a) The Euler angle parametrization becomes singular **(b)** $\dot{\psi} = q$ (c) $\dot{\phi} = r$ (d) $\dot{\phi} = \dot{\theta} = \dot{\psi} = 0$

2. The stability axes are a body-fixed reference frame defined relative to a particular wings-level equilibrium flight condition. The frame is defined such that *in nominal flight*

   (a) θ **is the climb angle** (b) $I_{xz} = 0$ (c) altitude is constant (d) $u = 0$

3. Recognizing that density decreases with altitude, which of the following statements describes what happens to the phugoid mode with increasing altitude, all other things staying equal? (*Hint:* How does density enter the approximate expressions for phugoid natural frequency and damping ratio?)

   (a) The natural frequency ω_{n_p} **decreases and the damping ratio** ζ_p **decreases.**
   (b) The natural frequency ω_{n_p} decreases and the damping ratio ζ_p increases.
   (c) The natural frequency ω_{n_p} increases and the damping ratio ζ_p increases.
   (d) The natural frequency ω_{n_p} increases and the damping ratio ζ_p decreases.

4. Which of the following statements is *not* true for a conventional airplane?

   (a) The roll moment $L_p p$ directly opposes rolling motion.
   (b) The roll moment $L_\beta \beta$ **directly opposes rolling motion.**
   (c) The pitch moment $M_q q$ directly opposes pitching motion.
   (d) The yaw moment $N_r r$ directly opposes yawing motion.

5. Which of the following actions will increase the roll mode time-to-half-amplitude (i.e., make the roll mode slower)?

   (a) Increasing the roll inertia I_x**.**
   (b) Increasing the wing span.
   (c) Moving the horizontal stabilizer farther aft.
   (d) Decreasing the magnitude of the initial roll rate.

6. The *neutral point* for an airplane is

   (a) the point at which the moment due to the aerodynamic force balances the aerodynamic moment.
   (b) the point about which the pitch moment does not vary with angle of attack.
   (c) the point at which the roll dynamics are neutrally stable.
   (d) the center of gravity.

7. If the CG is shifted forward in a statically stable, conventional aircraft, the elevator's effectiveness at changing the trim angle of attack

   (a) remains the same, (b) becomes greater, **(c) becomes smaller**, (d) vanishes entirely.

8. For an aircraft with a constant power propulsion system,

   (a) u remains constant. (b) α remains constant. (c) $\frac{\partial}{\partial u} T \big|_0 = 0$. **(d)** $\frac{\partial}{\partial u}(Tu)\big|_0 = 0$.

9. Which of the following design actions will make the ailerons more effective at exerting roll moments?

   (a) shortening each aileron's effective span.
   (b) moving the ailerons farther outward from the fuselage.
   (c) shortening the aileron chord relative to the wing chord.
   (d) decreasing the rudder size.

10. Which of the following statements is *not* generally true for conventional aircraft?

   (a) The phugoid mode evolves more slowly than the short period mode.
   (b) The Dutch roll mode evolves more slowly than the rolling convergence mode.
   (c) The rolling convergence mode evolves more slowly than the spiral mode.
   (d) The short period mode is stable.

11. Which of the following statements is *not* a standard assumption in formulating the equations of motion for an aircraft

   (a) X does not depend on q or $\dot{\alpha}$.
   (b) L does not depend on r and N does not depend on p.
   (c) Y, L, and N do not depend on θ, u, w, or q.
   (d) X, Z, and M do not depend on ϕ, ψ, v, p, or r.

12. Which of the following statements is *not* true? Closed-loop control ...

   (a) ... can stabilize an unstable plant.
   (b) ... can attenuate disturbances.
   (c) ... can mitigate sensitivity to plant parameter variations.
   (d) ... requires no sensors or other additional hardware.

13. The Routh-Hurwitz method is

   (a) a method for computing the circulation over a thin airfoil.
   (b) a method for determining the number of roots with positive real part for a given polynomial.
   (c) a method for determining the stick-fixed static margin from wind tunnel data.
   (d) a method for sizing an aircraft's control surfaces.

14. In classical control theory, the acronym "PID" represents

   (a) "proportional-integral-derivative" (a feedback control scheme)
   (b) "Pontryagin-Isidori-Doyle" (a famous stability theorem)
   (c) "position independent device" (a type of actuator)
   (d) "Pretty involved details"

15. The terms $Z_{\dot{\alpha}}$ and $M_{\dot{\alpha}}$ are primarily due to

   (a) the apparent mass of the wing and horizontal tail in the vertical direction.
   (b) an aeroelastic effect involving vertical oscillations of the wing.
   (c) second order effects of the propulsion system.
   (d) a time delay between changes in wing lift and their effect on downwash at the tail.