AOE 3134
Stability and Control
Final Exam

Name (10 points):

Part 1 Instructions: Solve *two* of the following three problems. If you attempt to solve all three, clearly mark the one which you do not wish to have graded.

Problem 1. (30 points) Consider an airplane with the following characteristics

$$\begin{aligned} C_L &= 0.09 + 0.09\,\bar{\alpha} + 0.003\,\delta e \\ C_m &= 0.06 - 0.03\,\bar{\alpha} - 0.01\,\delta e \end{aligned}$$

where all angles are measured in *degrees*. (Note: The overbar on $\bar{\alpha}$ signifies that angle of attack is not measured from the zero-lift line.) These equations are valid below stall. The lift coefficient at stall is $C_{L_{\max}} = 1.45$. The center of gravity is located at the point $x_{\text{cg}} = 0.5$ meters, where x is measured positive aft from the wing apex. Other relevant parameters are

$$W = 120 \text{ kN}, \qquad S = 30 \text{ m}^2, \qquad \bar{c} = 3 \text{ m}, \qquad b = 10 \text{ m}.$$

A. (10 points) Determine the equilibrium angle of attack and elevator deflection corresponding to steady, wings-level flight at speed $V = 100$ m/s in air of density $\rho = 1$ kg/m^3.

B. (10 points) Determine the stick-fixed neutral point location x_n.

C. (10 points) Suppose that elevator deflections are limited to the range $-15° \leq \delta e \leq 15°$. Find a forward limit $x_{\min}$ for the CG location defined by requiring that the minimum equilibrium elevator deflection $\delta e_{\min}$ correspond to the stall lift coefficient $C_{L_{\max}}$. Is this condition satisfied by the stated CG location?

Solution. (*A.*) The equilibrium flight conditions are

$$\begin{aligned} C_w &= C_{L_0} + C_{L_\alpha}\bar{\alpha} + C_{L_{\delta e}}\delta e \\ 0 &= C_{m_0} + C_{m_\alpha}\bar{\alpha} + C_{m_{\delta e}}\delta e \end{aligned}$$

where

$$\begin{aligned} C_w &= \frac{W}{\left(\frac{1}{2}\rho V^2\right) S} \\ &= 0.8 \end{aligned}$$

Solving, we find that

$$\begin{aligned} \begin{pmatrix} \bar{\alpha} \\ \delta e \end{pmatrix} &= \begin{pmatrix} C_{L_\alpha} & C_{L_{\delta e}} \\ C_{m_\alpha} & C_{m_{\delta e}} \end{pmatrix}^{-1} \begin{pmatrix} C_w - C_{L_0} \\ -C_{m_0} \end{pmatrix} \\ &= \begin{pmatrix} 8.5° \\ -19.6° \end{pmatrix}. \end{aligned}$$

$(B.)$ Since

$$K_n = h_n - h = -\frac{C_{m_\alpha}}{C_{L_\alpha}},$$

the neutral point is located at

$$\begin{aligned} x_n &= x_{cg} - \frac{C_{m_\alpha}}{C_{L_\alpha}} \\ &= \left(0.5 - \frac{-0.03}{0.09}(3)\right) \text{ m} \\ &= 1.5 \text{ m} \end{aligned}$$

$(C.)$ We have

$$\begin{aligned} h_{\min} &= h_n - \frac{C_{m_0} + C_{m_{\delta e}}\delta e_{\min}}{(C_{L_{\max}} - C_{L_0}) - C_{L_{\delta e}}\delta e_{\min}} \\ &= \frac{1}{2} - \frac{(0.06) + (-0.01)(-15)}{(1.45 - 0.09) - (0.003)(-15)} \\ &\approx 0.35 \end{aligned}$$

so that $x_{\min}$ is about 1.05 m. The condition *is not* satisfied since $x_{\text{cg}} < x_{\min}$.

Problem 2. (30 points) The inertial parameters for a certain airplane are

$$W = 17,600 \text{ lb}, \quad I_x = 8000 \text{ slug-ft}^2, \quad I_y = 26,000 \text{ slug-ft}^2, \quad I_z = 29,000 \text{ slug-ft}^2, \quad I_{xz} = 1300 \text{ slug-ft}^2$$

and the geometric parameters are

$$S = 260 \text{ ft}^2, \qquad b = 28 \text{ ft}, \qquad \bar{c} = 11 \text{ ft}.$$

The table below shows the lateral-directional stability coefficients corresponding to sea level flight at airspeed $V = 450$ ft/s. The density at sea level is $\rho \approx 0.0024$ slugs/ft^3. Where relevant, angles are in *radians.*

	$C_{Y_{(\cdot)}}$	$C_{l_{(\cdot)}}$	$C_{n_{(\cdot)}}$
β	-0.98	-0.12	0.25
p	0	-0.26	0.022
r	0	0.14	-0.35
δa	0	0.08	0.06
δr	0.17	-0.105	-0.32

A. (15 points) Compute the control deflections δa and δr required, and the corresponding roll angle ϕ, for this airplane to maintain heading given a 25 ft/s cross-wind coming from the right. (Assume that the total airspeed is the nominal value, $V = 450$ ft/s.)

B. (15 points) The roll rate is well-approximated by the first order equation

$$I_x \dot{p} = \mathcal{L}_p p + \mathcal{L}_{\delta a} \delta a$$

where

$$\mathcal{L}_{(\cdot)} = \frac{I_x}{I_x I_z - I_{xz}^2} \left(I_x L_{(\cdot)} + I_{xz} N_{(\cdot)} \right).$$

Use this equation to compute the approximate roll rate response $p(t)$ to a step aileron input

$$\delta a(t) = 0.05 \text{ rad}.$$

with $p(0) = 0$. *Suggestion:* You may find it easier to solve the for the step response symbolically and then substitute values.

Solution. (*A.*) From the equilibrium requirements

$$\begin{aligned} C_{Y_\beta}\beta + C_{Y_{\delta a}}\delta a + C_{Y_{\delta r}}\delta r + C_w \phi &= 0 \\ C_{l_\beta}\beta + C_{l_{\delta a}}\delta a + C_{l_{\delta r}}\delta r &= 0 \\ C_{n_\beta}\beta + C_{n_{\delta a}}\delta a + C_{n_{\delta r}}\delta r &= 0, \end{aligned}$$

we find that we need

$$\begin{pmatrix} C_{Y_{\delta a}} & C_{Y_{\delta r}} & C_w \\ C_{l_{\delta a}} & C_{l_{\delta r}} & 0 \\ C_{n_{\delta a}} & C_{n_{\delta r}} & 0 \end{pmatrix} \begin{pmatrix} \delta a \\ \delta r \\ \phi \end{pmatrix} = - \begin{pmatrix} C_{Y_\beta} \\ C_{l_\beta} \\ C_{n_\beta} \end{pmatrix} \beta$$

where

$$\beta = \sin^{-1}\left(\frac{\text{Crosswind velocity}}{\text{Total airspeed}} \right) = 0.056 \text{ rad}$$

or about 3.2°, and

$$C_w = \frac{W}{QS} = \frac{17,600}{\left(\frac{1}{2}(0.0024)(450)^2\right)(260)} = 0.28$$

Substituting parameter values,

$$\begin{pmatrix} 0 & 0.17 & 0.28 \\ 0.08 & -0.105 & 0 \\ 0.06 & -0.32 & 0 \end{pmatrix} \begin{pmatrix} \delta a \\ \delta r \\ \phi \end{pmatrix} = - \begin{pmatrix} -0.98 \\ -0.12 \\ 0.25 \end{pmatrix} (0.056).$$

Solving, we find

$$\delta a = 0.186 \text{ rad} \approx 10.7^\circ, \qquad \delta r = 0.078 \text{ rad} \approx 4.5^\circ, \qquad \phi = 0.147 \text{ rad} \approx 8.4^\circ.$$

$(B.)$ Solving the first order system first, with $\delta a = \bar{\delta a} 1(\tau)$, we find that

$$\begin{aligned} p(t) &= p(0) e^{\frac{\mathcal{L}_p}{I_x} t} + \int_0^t e^{\frac{\mathcal{L}_p}{I_x}\tau} \frac{\mathcal{L}_{\delta a}}{I_x} \bar{\delta a} 1(\tau) d\tau \\ &= \bar{\delta a} \frac{\mathcal{L}_{\delta a}}{\mathcal{L}_p} \left(e^{\frac{\mathcal{L}_p}{I_x} t} - 1 \right) \end{aligned}$$

We next compute

$$L_p = \frac{1}{4} \rho u_0 b^2 S C_{l_p} = -14,300 \text{ ft-lb/rad/s}$$

and

$$N_p = \frac{1}{4} \rho u_0 b^2 S C_{n_p} = 1200 \text{ ft-lb/rad/s}$$

so that

$$\begin{aligned} \mathcal{L}_p &= \frac{I_x}{I_x I_z - I_{xz}^2} \left(I_x L_p + I_{xz} N_p \right) \\ &= -3900 \text{ ft-lb/rad/s.} \end{aligned}$$

Next,

$$L_{\delta a} = \frac{1}{2} \rho u_0^2 b S C_{l_{\delta a}} = 142,000 \text{ ft-lb/rad}$$

and

$$N_{\delta a} = \frac{1}{2} \rho u_0^2 b S C_{n_{\delta a}} = 106,000 \text{ ft-lb/rad}$$

so that

$$\begin{aligned} \mathcal{L}_{\delta a} &= \frac{I_x}{I_x I_z - I_{xz}^2} \left(I_x L_{\delta a} + I_{xz} N_{\delta a} \right) \\ &= 44,000 \text{ ft-lb/rad.} \end{aligned}$$

Substituting values into

$$p(t) = \bar{\delta a} \frac{\mathcal{L}_{\delta a}}{\mathcal{L}_p} \left(e^{-\frac{\mathcal{L}_p}{I_x} t} - 1 \right)$$

gives

$$\begin{aligned} p(t) &= (0.05) \frac{44,000}{-3900} \left(e^{\frac{-3900}{8000} t} - 1 \right) \\ &= 0.56 \left(1 - e^{-0.49 t} \right) \text{ rad/s.} \end{aligned}$$

Problem 3. (30 points) Figure 1 depicts a simple model of Watt's regulator, an early mechanical feedback control device. The apparatus consists of a planar pendulum which rotates about the vertical axis. A simple dynamic model is

$$\ddot{\theta} + b\dot{\theta} + \sin\theta\left(\omega_c^2 - \omega^2\cos\theta\right) = 0$$

where θ is the elevation angle between the pendulum and the vertical axis, ω is the vertical component of angular rate, and $b > 0$ and ω_c are real-valued, constant parameters.

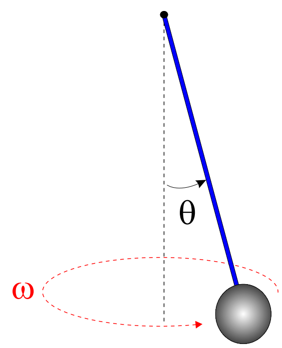

Figure 1: A simple model for Watt's regulator.

Assume that the shaft speed is controlled directly so that $u = \omega^2$ is an input. The equation above may be rewritten in first order form as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -\sin x_1\left(\omega_c^2 - u\cos x_1\right) - bx_2 \end{pmatrix}.$$

A. (15 points) Linearize the dynamics about the equilibrium $(\boldsymbol{x}, u) = \left([0,\ 0]^T, 2\omega_c^2\right)$. You should obtain equations in the form

$$\boldsymbol{\Delta}\dot{\boldsymbol{x}} = \boldsymbol{A}_{\mathrm{A}}\boldsymbol{\Delta x} + \boldsymbol{B}_{\mathrm{A}}\Delta u.$$

Show that the equilibrium is unstable by computing the eigenvalues of $\boldsymbol{A}_{\mathrm{A}}$.

B. (15 points) Linearize the dynamics about the equilibrium $(\boldsymbol{x}, u) = \left(\left[\frac{\pi}{3},\ 0\right]^T, 2\omega_c^2\right)$. You should obtain equations in the form

$$\boldsymbol{\Delta}\dot{\boldsymbol{x}} = \boldsymbol{A}_{\mathrm{B}}\boldsymbol{\Delta x} + \boldsymbol{B}_{\mathrm{B}}\Delta u.$$

Show that the equilibrium is stable by computing the eigenvalues of $\boldsymbol{A}_{\mathrm{B}}$.

Solution. (*A.*) Linearizing gives

$$\boldsymbol{A}_{\mathrm{A}} = \begin{pmatrix} 0 & 1 \\ \left.\frac{\partial f_2}{\partial x_1}\right|_e & -b \end{pmatrix}$$

where

$$\begin{aligned} \left.\frac{\partial f_2}{\partial x_1}\right|_e &= \left[\frac{\partial}{\partial x_1}\left(-\sin x_1\left(\omega_c^2 - u\cos x_1\right) - bx_2\right)\right]_e \\ &= \left[-\cos x_1\left(\omega_c^2 - u\cos x_1\right) - \sin x_1\left(u\sin x_1\right)\right]_e \\ &= -\left(\omega_c^2 - (2\omega_c^2)\right) \\ &= \omega_c^2. \end{aligned}$$

The input matrix is

$$\boldsymbol{B}_{\mathrm{A}} = \begin{pmatrix} 0 \\ \left.\frac{\partial f_2}{\partial u}\right|_e \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic polynomial for the state matrix $\boldsymbol{A}_{\mathrm{A}}$ is

$$\|\lambda\mathbb{I} - \boldsymbol{A}_{\mathrm{A}}\| = \det\begin{pmatrix} \lambda & -1 \\ -\omega_c^2 & \lambda + b \end{pmatrix} = \lambda^2 + b\lambda - \omega_c^2.$$

The roots are

$$\lambda_{1,2} = \frac{1}{2}\left(-b \pm \sqrt{b^2 + 4\omega_c^2}\right).$$

Note that one of these roots is positive – the equilibrium at the origin is unstable.

(B.) Linearizing gives

$$\boldsymbol{A}_{\mathrm{B}} = \begin{pmatrix} 0 & 1 \\ \left.\frac{\partial f_2}{\partial x_1}\right|_e & -b \end{pmatrix}$$

where

$$\begin{aligned}
\left.\frac{\partial f_2}{\partial x_1}\right|_e &= \left[\frac{\partial}{\partial x_1}\left(-\sin x_1\left(\omega_c^2 - u\cos x_1\right) - bx_2\right)\right]_e \\
&= \left[-\cos x_1\left(\omega_c^2 - u\cos x_1\right) - \sin x_1\left(u\sin x_1\right)\right]_e \\
&= -\cos\left(\frac{\pi}{3}\right)\left(\omega_c^2 - (2\omega_c^2)\cos\left(\frac{\pi}{3}\right)\right) - \sin\left(\frac{\pi}{3}\right)\left((2\omega_c^2)\sin\left(\frac{\pi}{3}\right)\right) \\
&= -\frac{1}{2}\omega_c^2\left(1 - 2\left(\frac{1}{2}\right)\right) - \left(\frac{\sqrt{3}}{2}\right)(2\omega_c^2)\left(\frac{\sqrt{3}}{2}\right) \\
&= -\frac{3}{2}\omega_c^2.
\end{aligned}$$

The input matrix is

$$\boldsymbol{B}_{\mathrm{B}} = \begin{pmatrix} 0 \\ \left.\frac{\partial f_2}{\partial u}\right|_e \end{pmatrix} = \begin{pmatrix} 0 \\ \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right) \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{\sqrt{3}}{4} \end{pmatrix}.$$

The characteristic polynomial for the state matrix $\boldsymbol{A}_{\mathrm{B}}$ is

$$\|\lambda\mathbb{I} - \boldsymbol{A}_{\mathrm{B}}\| = \det\begin{pmatrix} \lambda & -1 \\ \frac{3}{2}\omega_c^2 & \lambda + b \end{pmatrix} = \lambda^2 + b\lambda + \frac{3}{2}\omega_c^2.$$

The roots

$$\lambda_{1,2} = \frac{1}{2}\left(-b \pm \sqrt{b^2 - 6\omega_c^2}\right)$$

always have negative real part.

Part 2 Instructions: Circle the letter corresponding to the correct answer for each of the following ten questions. Each question is worth three points. (Don't forget to turn these answers in with your solutions to Part 1!)

1. The state $\boldsymbol{x} = \mathbf{0}$ is a stable equilibrium of the LTI system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$ provided

(a) the initial state is $\boldsymbol{x}(0) = \mathbf{0}$.
(b) there are no complex conjugate eigenvalue pairs.
(c) at least one eigenvalue of $\boldsymbol{A}$ has positive real part.
(d) every eigenvalue of $\boldsymbol{A}$ has negative real part.

2. The term $C_{n_{\delta a}}$ arises from

(a) the secondary roll-stabilizing effect of wing dihedral.
(b) differential drag on the left and right wing due to an aileron deflection.
(c) a gyroscopic moment due to rapid roll accelerations.
(d) a change in wing-body interference caused by aileron deflections.

3. The *center of pressure* is

(a) the point at which the moment due to the aerodynamic force generated by a wing precisely balances the aerodynamic moment.
(b) the point about which the pitch moment does not vary with angle of attack.
(c) the point at which the buoyant force acts.
(d) a facility where world-renowned scientists study the many intriguing effects of pressure.

4. Which of the following statements is *false*?

(a) For a wing in a wind tunnel, $C_{m_{0L}} = C_{m_{\text{ac}}}$.
(b) An airplane is statically stable in pitch provided $C_{m_0} > 0$ and $C_{m_\alpha} < 0$.
(c) A forward horizontal tail (a canard) provides a negative increment in C_{m_α}.
(d) To first order, the tail incidence angle i_t affects C_{m_0} but not C_{m_α}.

5. For a stable aircraft with reversible controls, the stick-free static margin is generally

(a) negative.
(b) smaller than the stick-fixed static margin.
(c) larger than the stick-fixed static margin.
(d) impossible to compute.

6. For a steady turn at constant radius, the pitch rate

(a) is zero.
(c) varies sinusoidally.
(b) depends on the roll and yaw rate.
(d) depends on load factor and speed.

7. Which of the following steps would increase the magnitude of C_{l_β}?

(a) Increasing the dihedral angle.
(b) Mounting the wing below the fuselage.
(c) Sweeping the wings forward.
(d) All of the above.

8. Linearization is

(a) a wing design technique that ensures lift is linear in angle of attack.
(b) an aircraft design technique used to make an airplane's dynamics linear.
(c) a navigation technique in which the airplane flies straight-line segments between way-points.
(d) a mathematical technique in which nonlinear ODEs are approximated by linear ODEs.

9. For small perturbations from a steady, but steep wings-level climb

(a) $\Delta\dot{\phi} = \Delta p$ **(b)** $\Delta\dot{\theta} = \Delta q$ (c) $\Delta\dot{\psi} = \Delta r$ (d) The linear approximation fails.

10. For typical aircraft, the short period mode is

(a) overdamped. (b) critically damped. **(c) underdamped.** (d) unstable.