AOE 3134
Stability and Control
Exam #2 Solutions

Problem 1. (40 points) Consider the following second order system:

$$\ddot{x} + b\left(e^{\dot{x}} - 1\right) + kx(1 - x^2) = 0$$

(a) Define $x_1 = x$ and $x_2 = \dot{x}$ and convert the second order equation into two first order equations.

(b) Find all of the equilibria.

(c) Linearize the first order equations about the equilibrium $(x_1, x_2)_{\text{eq}} = (0, 0)$.

(d) Let $k = 5$ and $b = 2$. Determine the time to half amplitude for the linearized dynamics.

Solution. With $x_1 = x$ and $x_2 = \dot{x}$, the single second order equation becomes

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -b\left(e^{x_2} - 1\right) - kx_1(1 - x_1^2). \end{aligned}$$

Equilibria are the constant values of $\boldsymbol{x} = [x_1, x_2]^T$ which satisfy the equations. Considering the first equation, we see that $x_2 = 0$ at equilibria. Substituting this value of x_2 into the second equation, we see that there are three possible equilibria:

$$\boldsymbol{x}_{\text{eq}} \in \left\{ \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \right\}$$

To linearize about the second of these equilibria, we define

$$\Delta\boldsymbol{x} = \boldsymbol{x} - \boldsymbol{x}_{\text{eq}}$$

and compute

$$\Delta\dot{\boldsymbol{x}} = \boldsymbol{A}\Delta\boldsymbol{x}$$

where $\boldsymbol{A}$ is the Jacobian of

$$\boldsymbol{f}(\boldsymbol{x}) = \begin{pmatrix} x_2 \\ -b\left(e^{x_2} - 1\right) - kx_1(1 - x_1^2) \end{pmatrix}$$

evaluated at the equilibrium. The Jacobian is

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} = \begin{pmatrix} 0 & 1 \\ -k\left(1 - 3x_1^2\right) & -be^{x_2} \end{pmatrix}.$$

Evaluating at the equilibrium gives

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ -k & -b \end{pmatrix}.$$

The characteristic values are the roots of

$$|\lambda\mathbb{I} - \boldsymbol{A}| = \lambda^2 + b\lambda + k$$

With $b = 2$ and $k = 5$, these values are

$$\begin{aligned}\lambda_{1,2} &= \frac{1}{2}\left(-2 \pm \sqrt{4-20}\right) \\ &= -1 \pm 2j\end{aligned}$$

Since the real part of the characteristic values is negative, the response of the linear system to initial conditions is oscillatory decay. To determine the time to half amplitude, note that

$$\zeta\omega_n = -\mathrm{Re}\left(\lambda_{1,2}\right) \qquad \text{and} \qquad \omega_d = \left|\mathrm{Im}\left(\lambda_{1,2}\right)\right|$$

The time to half amplitude is

$$t_{\text{half}} = \frac{0.69}{\zeta\omega_n} = 0.69 \text{ s}$$

Problem 2. (35 points) Consider an aircraft with the following parameters:

$$W = 2750 \text{ lb} \quad (m = 85.4 \text{ slug}), \quad I_{\text{x}} = 1050 \text{ slug ft}^2, \quad I_{\text{z}} = 3530 \text{ slug ft}^2, \quad I_{\text{xz}} = 0 \text{ slug ft}^2$$

and

$$S = 184 \text{ ft}^2, \quad b = 33 \text{ ft.}$$

For sea level flight at speed 176 ft/s, the relevant stability derivatives are

	$Y_{(\cdot)}$	$L_{(\cdot)}$	$N_{(\cdot)}$
v	-21.8 lb/(ft/s)	-94.2 (ft lb)/(ft/s)	90.4 (ft lb)/(ft/s)
p	0 lb/(rad/s)	$-8.61 \cdot 10^3$ (ft lb)/(rad/s)	$-1.21 \cdot 10^3$ (ft lb)/(rad/s)
r	218.8 lb/(rad/s)	$2.25 \cdot 10^3$ (ft lb)/(rad/s)	$-2.63 \cdot 10^3$ (ft lb)/(rad/s)

(a) Approximate the spiral and rolling convergence eigenvalues and the Dutch roll natural frequency and damping ratio.

(b) Discuss, in *complete* sentences, the effect that increasing the vertical tail volume ratio would have on the Dutch roll natural frequency for this aircraft. (Be specific. Consider how changes in vertical tail volume ratio affect the values of the relevant stability derivatives in the dutch roll approximation.)

Solution. Starting with Part a), we have the following approximations.

Spiral mode:

$$\begin{aligned}\tilde{\lambda}_{\text{s}} &= \frac{\left(\mathcal{L}_r\mathcal{N}_v - \mathcal{L}_v\mathcal{N}_r\right)}{-I_z\mathcal{L}_v + \frac{u_0}{g}\left(\mathcal{L}_v\mathcal{N}_p - \mathcal{L}_p\mathcal{N}_v\right)} \\ &= \frac{\left(L_rN_v - L_vN_r\right)}{-I_zL_v + \frac{u_0}{g}\left(L_vN_p - L_pN_v\right)} \\ &= -0.0083\end{aligned}$$

Roll mode:

$$\begin{aligned}\tilde{\lambda}_{\text{r}} &= \frac{1}{I_x}\mathcal{L}_p \\ &= \frac{1}{I_x}L_p \\ &= -8.2\end{aligned}$$

Dutch roll mode:

$$\begin{aligned}
\tilde{\omega}_{n_{\mathrm{dr}}} &= \sqrt{\frac{1}{mI_z}\left(Y_v\mathcal{N}_r - \mathcal{N}_v Y_r\right) + \frac{u_0}{I_z}\mathcal{N}_v} \\
&= \sqrt{\frac{1}{mI_z}\left(Y_v N_r - N_v Y_r\right) + \frac{u_0}{I_z}N_v} \\
&= 2.15 \text{ rad/s}
\end{aligned}$$

$$\begin{aligned}
\tilde{\zeta}_{\mathrm{dr}} &= -\frac{1}{2\omega_{n_{\mathrm{dr}}}}\left(\frac{1}{m}Y_v + \frac{1}{I_z}\mathcal{N}_r\right) \\
&= -\frac{1}{2\omega_{n_{\mathrm{dr}}}}\left(\frac{1}{m}Y_v + \frac{1}{I_z}N_r\right) \\
&= 0.23
\end{aligned}$$

Turning to Part b), notice that the dutch roll natural frequency and damping ratio depend on tail size and location in several ways. Inertial properties such as mass m and yaw inertia I_z certainly may vary with tail volume ratio. These effects are almost certainly dominated by the aerodynamic effects, however; why else would one change the tail volume ratio? The stability derivatives Y_v, Y_r, N_v, and N_r all vary with tail volume ratio. An increase in vertical tail size increases the magnitude of all four terms and an increase in distance to the vertical tail increases the magnitude of N_v and N_r. Noting that the first term dominates in the sum $Y_v N_r - N_v Y_r$ and that the N_v term is positive, one should expect an increase in vertical tail volume ratio to increase $\tilde{\omega}_{n_{\mathrm{dr}}}$.

Problem 3. (30 points) For each of the following questions, circle the letter corresponding to the best answer. Each question is worth six points. (Don't forget to turn these answers in with your solutions to Problems 1 and 2!)

1. Suppose that, at a given instant, a certain airplane's pitch angle θ passes through 0° and its roll angle ϕ passes through 90°. Then, at this same instant,

(a) $\dot{\psi} = q$ (b) $\dot{\phi} = r$ (c) $\dot{\phi} = \dot{\theta} = \dot{\psi} = 0$ (d) The Euler angle parametrization becomes singular

2. The stability axes are a body-fixed reference frame defined relative to a particular wings-level equilibrium flight condition. The frame is defined such that *in nominal flight*

(a) $I_{\text{xz}} = 0$ (b) $u = 0$ (c) altitude is constant **(d) θ is the climb angle**

3. Recognizing that density decreases with increasing altitude, which of the following statements describes what happens to the phugoid mode with increasing altitude (all other things equal)

(a) The natural frequency ω_{n_p} increases and the damping ratio ζ_p increases.
(b) The natural frequency ω_{n_p} increases and the damping ratio ζ_p decreases.
(c) The natural frequency ω_{n_p} decreases and the damping ratio ζ_p decreases.
(d) The natural frequency ω_{n_p} decreases and the damping ratio ζ_p increases.

4. Which of the following statements is *not* true for a conventional airplane?

(a) The roll moment $L_p p$ directly opposes rolling motion.
(b) The roll moment $L_\beta \beta$ directly opposes rolling motion.
(c) The pitch moment $M_q q$ directly opposes pitching motion.
(d) The yaw moment $N_r r$ directly opposes yawing motion.

5. Which of the following will *not* decrease the time to half amplitude for the roll mode, given some nonzero initial roll rate?

(a) Decreasing the roll inertia I_x.
(b) Increasing the wing span.
(c) Increasing the speed.
(d) Decreasing the magnitude of the initial roll rate.