

AOE 3134
Stability and Control
Exam #2 Solutions

Problem 1. (30 points) The following second order system describes an oscillator with a stiffening spring and nonlinear damping:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -kx_1(1 + x_1^2) - b \sin(x_2) \end{pmatrix}. \quad (1)$$

(a) Find all equilibria.

(b) Linearize equations (1) about the equilibrium $(x_1, x_2)_{\text{eq}} = (0, 0)$.

Bonus (10 points) Define $x = x_1$ and note that $x_2 = \dot{x}$. Rewrite your solution to part (b) as a single, second order ODE in terms of the dependent variable x and its derivatives. Let

$$k = 2 \quad \text{and} \quad b = 3$$

Determine the response $x(t)$ to an arbitrary initial state $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$.

Solution. Equilibria satisfy

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x_2 \\ -kx_1(1 + x_1^2) - b \sin(x_2) \end{pmatrix}.$$

The first equation gives $x_2|_{\text{eq}} = 0$ and the second then gives

$$[x_1(1 + x_1^2)]_{\text{eq}} = 0.$$

We therefore have only one possible equilibrium:

$$(x_1, x_2)_{\text{eq}} = (0, 0)$$

(The dependent variables x_1 and x_2 must be real-valued.) Linearizing about this equilibrium, and noting that $\Delta x_i = x_i$ for this equilibrium, we find

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

where

$$\begin{aligned} \mathbf{A} = \left. \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right|_0 &= \begin{pmatrix} 0 & 1 \\ -k(1 + 3x_1^2) & -b \cos(x_2) \end{pmatrix}_0 \\ &= \begin{pmatrix} 0 & 1 \\ -k & -b \end{pmatrix}. \end{aligned}$$

With $x = x_1$ and $\dot{x} = x_2$, we may write

$$\dot{x}_2 = -kx_1 - bx_2 \quad \rightarrow \quad \ddot{x} = -k\dot{x} - bx$$

or

$$\ddot{x} + b\dot{x} + kx = 0.$$

The roots of the characteristic equation

$$\lambda^2 + b\lambda + k = 0$$

are

$$\begin{aligned}\lambda_{1,2} &= \frac{1}{2} \left(-b \pm \sqrt{b^2 - 4k} \right) \\ &= \frac{1}{2} \left(-3 \pm \sqrt{9 - 8} \right) \\ &= -1, -2\end{aligned}$$

The general solution is therefore

$$x(t) = c_1 e^{-t} + c_2 e^{-2t}.$$

Evaluating this expression and its derivative at the initial time gives

$$x(0) = x_0 = c_1 + c_2 \quad \text{and} \quad \dot{x}(0) = \dot{x}_0 = -c_1 - 2c_2.$$

In matrix form,

$$\begin{pmatrix} x_0 \\ \dot{x}_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

The coefficients c_1 and c_2 are thus given by

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}^{-1} \begin{pmatrix} x_0 \\ \dot{x}_0 \end{pmatrix} = - \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ \dot{x}_0 \end{pmatrix} = \begin{pmatrix} 2x_0 + \dot{x}_0 \\ -x_0 - \dot{x}_0 \end{pmatrix}.$$

Thus, in the end, we find that

$$x(t) = (2x_0 + \dot{x}_0) e^{-t} - (x_0 + \dot{x}_0) e^{-2t}.$$

Problem 2. (40 points) Table 1 gives the longitudinal stability derivatives for a certain four-engine, transport jet flying horizontally at Mach 0.25 at sea level conditions.¹ Other relevant parameters include:

$$W = 126,000 \text{ lbs} \quad S = 2,000 \text{ ft}^2 \quad \bar{c} = 11 \text{ ft} \quad I_y = 2,450,000 \text{ slug} - \text{ft}^2.$$

Table 1: Longitudinal Stability Derivatives.

	$C_{D(\cdot)}$	$C_{L(\cdot)}$	$C_{m(\cdot)}$
0 (nominal)	0.08	0.68	0
α	0.3	4.5	-0.9
$\dot{\alpha}$	0	2.7	-4.1
q	0	7.7	-12.1
Ma	0	0	0

(a) Compute approximate values for the natural frequency and damping ratio of the short period mode. Use your approximation to estimate the time and number of cycles to half amplitude.

¹For the standard atmosphere at sea level, density is $\rho = 2.3769 \times 10^{-3}$ slugs/ft³ and the speed of sound is $a = 1116.45$ ft/s.

(b) Compute approximate values for the natural frequency and damping ratio of the phugoid mode. Use your approximation to estimate the time and number of cycles to half amplitude.

Solution. The approximate short period natural frequency and damping ratio are

$$\omega_{n_{sp}} \approx \sqrt{\frac{1}{mI_y} Z_w M_q - \frac{u_0}{I_y} M_w} \quad \text{and} \quad \zeta_{sp} \approx -\frac{1}{2\omega_{n_{sp}}} \left[\frac{1}{m} Z_w + \frac{1}{I_y} (M_q + u_0 M_{\dot{w}}) \right].$$

For a jet, the phugoid natural frequency and damping ratio may be crudely estimated as

$$\omega_{n_p} \approx \sqrt{2} \frac{g}{u_0} \quad \text{and} \quad \zeta_p \approx \frac{1}{\sqrt{2}} \frac{C_D}{C_L}.$$

A more accurate approximation is

$$\omega_{n_p} \approx \sqrt{-\frac{Z_u g}{m u_0}} \quad \text{and} \quad \zeta_p \approx -\frac{X_u}{2m\omega_{n_p}}.$$

The nominal flight speed is

$$u_0 = \text{Ma } a = 279 \text{ ft/s}$$

and the vehicle mass is

$$m = \frac{W}{g} = 3913 \text{ slugs.}$$

In addition to the speed and mass, we need the following dimensional stability derivatives:

$$X_u, \quad Z_u, \quad Z_w, \quad M_q, \quad M_w, \quad \text{and} \quad M_{\dot{w}}.$$

Table 2: Longitudinal Dimensional Stability Derivatives

	$X_{(\cdot)}$	$Z_{(\cdot)}$	$M_{(\cdot)}$
u	$\frac{1}{2}\rho u_0 S [2(-C_{D_0} + C_{T_0}) + (-C_{D_u} + C_{T_u})]$	$-\frac{1}{2}\rho u_0 S (2C_{L_0} + C_{L_u})$	$\frac{1}{2}\rho u_0 S \bar{c} C_{m_u}$
w	$\frac{1}{2}\rho u_0 S (-C_{D_\alpha} + C_{L_0})$	$-\frac{1}{2}\rho u_0 S (C_{D_0} + C_{L_\alpha})$	$\frac{1}{2}\rho u_0 S \bar{c} C_{m_\alpha}$
q	≈ 0	$-\frac{1}{4}\rho u_0 S \bar{c} C_{L_q}$	$\frac{1}{4}\rho u_0 S \bar{c}^2 C_{m_q}$
\dot{w}	≈ 0	$-\frac{1}{4}\rho S \bar{c} C_{L_{\dot{\alpha}}}$	$\frac{1}{4}\rho S \bar{c}^2 C_{m_{\dot{\alpha}}}$

The only terms in Table 2 which are not given explicitly in the problem statement are

$$C_{T_0}, \quad C_{T_u}, \quad C_{D_u}, \quad C_{L_u}, \quad \text{and} \quad C_{m_u}.$$

Assuming that the latter three terms depend only on compressibility effects, we find that

$$\begin{aligned} C_{D_u} &= C_{D_{\text{Ma}}} \text{Ma} = 0 \\ C_{L_u} &= C_{L_{\text{Ma}}} \text{Ma} = 0 \\ C_{m_u} &= C_{m_{\text{Ma}}} \text{Ma} = 0. \end{aligned}$$

We are left, then, with C_{T_0} and C_{T_u} . First, we note that

$$-C_{D_0} + C_{T_0} = C_{w_0} \sin \theta_0 = 0 \quad \Rightarrow \quad C_{T_0} = C_{D_0}.$$

Next, recognizing that this is a jet aircraft, we assume *constant thrust flight* so that

$$C_{T_u} = -2C_{T_0} = -2C_{D_0}.$$

In the end, we find the following values for the dimensional stability derivatives

$$\begin{aligned}
 X_u &= \frac{1}{2}\rho u_0 S [2(-C_{D_0} + C_{T_0}) + (-C_{D_u} + C_{T_u})] = -106 \frac{\text{lb}}{\text{ft/s}} \\
 Z_u &= -\frac{1}{2}\rho u_0 S (2C_{L_0} + C_{L_u}) = -903 \frac{\text{lb}}{\text{ft/s}} \\
 Z_w &= -\frac{1}{2}\rho u_0 S (C_{D_0} + C_{L_\alpha}) = -3,039 \frac{\text{lb}}{\text{ft/s}} \\
 M_q &= \frac{1}{4}\rho u_0 S \bar{c}^2 C_{m_q} = -4.857 \times 10^5 \frac{\text{ft lb}}{\text{rad/s}} \\
 M_w &= \frac{1}{2}\rho u_0 S \bar{c} C_{m_\alpha} = -6.568 \times 10^3 \frac{\text{ft lb}}{\text{ft/s}} \\
 M_{\dot{w}} &= \frac{1}{4}\rho S \bar{c}^2 C_{m_{\dot{\alpha}}} = -590 \frac{\text{ft lb}}{\text{ft/s}^2}
 \end{aligned}$$

Substituting values into the approximations above gives

$$\omega_{n_{\text{sp}}} \approx 0.95 \text{ rad/s} \quad \text{and} \quad \zeta_{\text{sp}} \approx 0.55$$

and

$$\omega_{n_{\text{p}}} \approx 0.16 \text{ rad/s} \quad \text{and} \quad \zeta_{\text{p}} \approx 0.083.$$

Notice how the phugoid natural frequency and damping ratio compare with the more crude approximation:

$$\omega_{n_{\text{p}}} \approx \sqrt{2} \frac{g}{u_0} = 0.16 \text{ rad/s} \quad \text{and} \quad \zeta_{\text{p}} \approx \frac{1}{\sqrt{2}} \frac{C_D}{C_L} = 0.083.$$

The results are *identical*, at least to the given accuracy.

The time and number of cycles to half amplitude for each mode are

$$t_{\text{half}_{\text{sp}}} = \frac{0.69}{\zeta_{\text{sp}} \omega_{n_{\text{sp}}}} \approx 1.33 \text{ s} \quad \text{and} \quad N_{\text{half}_{\text{sp}}} = 0.11 \sqrt{\frac{1 - \zeta_{\text{sp}}^2}{\zeta_{\text{sp}}^2}} \approx 0.17$$

and

$$t_{\text{half}_{\text{p}}} = \frac{0.69}{\zeta_{\text{p}} \omega_{n_{\text{p}}}} \approx 51 \text{ s} \quad \text{and} \quad N_{\text{half}_{\text{p}}} = 0.11 \sqrt{\frac{1 - \zeta_{\text{p}}^2}{\zeta_{\text{p}}^2}} \approx 1.3$$

Problem 3. (30 points) For each of the following questions, circle the letter corresponding to the correct answer. Each question is worth five points. (Don't forget to turn these answers in with your solutions to Problems 1 and 2!)

1. Suppose that, at a given instant, a certain airplane's pitch angle θ passes through 90° . Then, at this same instant,

- (a) the body angular velocity ω becomes undefined.
- (b) the matrix \mathbf{R}_{IB} which rotates body frame vectors into the inertial frame becomes singular.
- (c) the matrix \mathbf{L}_{BI} which relates ω to the Euler angle rates becomes singular.**
- (d) the airplane explodes.

2. The stability axes are a body-fixed reference frame defined relative to a particular wings-level equilibrium flight condition. The frame is defined such that, in nominal flight,

- (a) $w = 0$** (b) $u = 0$ (c) $z = 0$ (d) $I_{xz} = 0$

3. Which of the following statements is *not* a standard assumption in formulating the equations of motion for an aircraft

- (a) Y , L , and N do not depend on θ , u , w , or q .
- (b) X , Z , and M do not depend on ϕ , ψ , v , p , or r .
- (c) N does not depend on v .**
- (d) X does not depend on q or \dot{w} .

4. The terms $Z_{\dot{w}}$ and $M_{\dot{w}}$ are primarily due to

- (a) the apparent mass of the wing and horizontal tail in the vertical direction.
- (b) an aeroelastic effect involving vertical oscillations of the wing.
- (c) second order effects of the propulsion system.
- (d) a time delay between changes in wing lift and their effect on downwash at the tail.**

5. For a constant thrust propulsion system,

- (a) $\left. \frac{\partial T}{\partial u} \right|_0 = 0$,** (b) $C_{T_u} = 0$, (c) $C_{X_u} = 0$, (d) $C_{L_\alpha} = 0$

6. For a conventional, dynamically stable airplane:

- (a) $C_{m_q} < 0$, $C_{n_r} > 0$, and $C_{l_p} < 0$.
- (b) $C_{m_q} < 0$, $C_{n_r} < 0$, and $C_{l_p} > 0$.
- (c) $C_{m_q} > 0$, $C_{n_r} > 0$, and $C_{l_p} > 0$.
- (d) $C_{m_q} < 0$, $C_{n_r} < 0$, and $C_{l_p} < 0$.**